MA99 EDUCATION

Mathematics Extension 2: Practice Paper 1

01

NSW MATHEMATICS EXTENSION 2 11-12 SYLLABUS 2024

FOR HSC 2027 AND LATER

General Instructions

- Reading time: 10 minutes
- Working time: 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11 to 16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I · 10 marks

Questions 1 to 10

Suggested time: about 15 minutes

Section II · 90 marks

Questions 11 to 16

Suggested time: about 2 hours 45 minutes

Mathematics Advanced and Mathematics Extension 1 are assumed knowledge for this examination.

Marks may be deducted for careless or badly arranged work.

Section I · Answer Sheet

Shade ONE circle for each of Questions 1 to 10. If you change your mind, draw a cross through the incorrect answer and shade the new one.

1	A	B	C	D	2	A	B	C	D
3	A	B	C	D	4	A	B	C	D
5	A	B	C	D	6	A	B	C	D
7	A	B	C	D	8	A	B	C	D
9	A	B	C	D	10	A	B	C	D

Section I · 10 marks

Answer Questions 1 to 10 on the answer sheet, spending about 15 minutes on this section.

Question 1

1 MARK

Which statement is the negation of "for every integer n, $n^2 + n$ is even"?

A For every integer n, $n^2 + n$ is odd.

B There exists an integer n for which $n^2 + n$ is odd.

C There exists an integer n for which $n^2 + n$ is even.

D For every integer n, $n^2 + n$ is not odd.

Question 2

1 MARK

Which value of n is a counterexample to the claim that $n^2 + n + 1$ is prime for every positive integer n?

A $n = 1$

B $n = 2$

C $n = 3$

D $n = 4$

Question 3

1 MARK

A line has vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Which point lies on the line?

A $(5, 0, 5)$

B $(5, 4, 1)$

C $(3, 3, 4)$

D $(2, -1, 1)$

Question 4

1 MARK

What is the angle between $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$?

A 30°

B 45°

C 60°

D 90°

Question 5

1 MARK

What is $\operatorname{Arg}\left(-1 - i\sqrt{3}\right)$?

A $\dfrac{2\pi}{3}$

B $\dfrac{4\pi}{3}$

C $-\dfrac{2\pi}{3}$

D $-\dfrac{\pi}{3}$

Question 6

1 MARK

What is $(1 + i)^6$?

A $8i$

B $-8i$

C 8

D -8

Question 7

1 MARK

What is $\int xe^x\,dx$?

A $xe^x + C$

B $(x-1)e^x + C$

C $(x+1)e^x + C$

D $\dfrac{x^2}{2}e^x + C$

Question 8

1 MARK

What is $\int \dfrac{2x}{x^2+1}\,dx$?

A $\ln(x^2+1) + C$

B $2\ln(x^2+1) + C$

C $\frac{1}{2}\ln(x^2+1) + C$

D $2\tan^{-1}x + C$

Question 9

1 MARK

A particle moves so that $\ddot{x} = -16x$. What is the period of the motion?

A $\dfrac{\pi}{8}$

B $\dfrac{\pi}{4}$

C $\dfrac{\pi}{2}$

D 8π

Question 10

1 MARK

A body falls from rest with $\ddot{x} = 10 - 0.2v$. What is its terminal velocity?

A 2

B $\sqrt{50}$

C 9.8

D 50

Section II · 90 marks

Answer Questions 11 to 16 in the spaces provided, spending about 2 hours 45 minutes on this section. Show your mathematical reasoning and/or calculations in every answer.

Question 11 · 15 marks

The nature of proof

(a) Prove by contradiction that $\sqrt{6}$ is irrational. 3 MARKS

(b) Prove by mathematical induction that $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ for every positive integer n. 3 MARKS

(c) Prove that $a + \dfrac{1}{a} \geq 2$ for every real $a > 0$. 2 MARKS

(d) State the value of a for which equality holds in part (c). 1 MARK

(e) Prove that $n^3 - n$ is divisible by 6 for every integer n. 3 MARKS

(f) The claim "$n^2 - n + 41$ is prime for every positive integer n" is false. Give a counterexample and justify it. 2 MARKS

(g) Explain why testing that claim for $n = 1$ to $n = 40$ does not prove it. 1 MARK

Question 12 · 15 marks

Further work with vectors

(a) Write down a vector equation for the line through $A(1, 1, 1)$ and $B(3, 4, 5)$. 2 MARKS

(b) Determine whether $C(7, 10, 13)$ lies on that line. 2 MARKS

(c) Show that $\ell_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\ell_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + u \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ are skew. 4 MARKS

(d) Show that $P(3, 3, 4)$ lies inside the sphere $\left| \mathbf{r} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right| = 6$. 2 MARKS

(e) Show that $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ are perpendicular. 2 MARKS

(f) $OABC$ is a parallelogram with $\vec{OA} = \mathbf{a}$ and $\vec{OC} = \mathbf{c}$. Prove that its diagonals are perpendicular if and only if $|\mathbf{a}| = |\mathbf{c}|$. 3 MARKS

Question 13 · 15 marks

Complex numbers and the complex plane

(a) Express $\dfrac{3+2i}{1-i}$ in the form $a+ib$. 2 MARKS

(b) Find the modulus and principal argument of $-2+2i$. 2 MARKS

(c) Use de Moivre's theorem to evaluate $\left(1+i\sqrt{3}\right)^6$. 3 MARKS

(d) Find the two square roots of $-8 + 6i$. 3 MARKS

(e) Prove that $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ for all complex z_1 and z_2. 2 MARKS

(f) Given $|z| = 1$ and $z \neq \pm 1$, prove that $\dfrac{1+z}{1-z}$ is purely imaginary. 2 MARKS

(g) State the condition for equality in the triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$. 1 MARK

Question 14 · 15 marks

Roots of complex numbers and loci

(a) Solve $z^4 = 16$ over $\mathbb{C}$, giving the roots in Cartesian form. 3 MARKS

(b) Solve $z^3 = -64i$ over $\mathbb{C}$, giving the roots in Cartesian form. 4 MARKS

(c) Describe the locus $|z - 2| = |z + 2i|$ and find its Cartesian equation. 2 MARKS

π

(d) Describe the locus $\arg(z-1)=\dfrac{\pi}{3}$, stating any excluded point. 2 MARKS

(e) Find the area of the region $\left\{z : |z| \leq 3 \text{ and } 0 \leq \arg z \leq \dfrac{\pi}{4}\right\}$. 2 MARKS

(f) Explain why the n solutions of $z^n = c$ sum to zero for every $n \geq 2$ and every non zero complex c. 2 MARKS

Question 15 · 15 marks

Further integration

(a) Find $\displaystyle\int \frac{x}{x^2+4}\,dx$. 2 MARKS

(b) Evaluate $\int_0^1 xe^{2x}\,dx$. 3 MARKS

(c) Let $I_n = \int_0^1 x^n e^{2x}\,dx$ for integers $n \geq 0$ Show that $2I_n = e^2 - nI_{n-1}$ for $n \geq 1$. 2 MARKS

(d) Given that $I_0 = \frac{1}{2}\left(e^2 - 1\right)$, use the relation in part (c) to evaluate $\int_0^1 x^2 e^{2x}\,dx$. 2 MARKS

(e) Evaluate $\int_0^{\pi/2} \sin 3x \cos x\,dx$. 3 MARKS

(f) A particle of mass m moves against a resistance mkv^2 with initial speed u. Use integration to show that $v = ue^{-kx}$, where x is the distance travelled. 3 MARKS

Question 16 · 15 marks

Applications of calculus to mechanics

(a) A particle moves so that $x = 5\sin 4t + 3$. Show that the motion is simple harmonic and state its centre, amplitude and period. 3 MARKS

(b) For the motion in part (a), find the maximum speed and the speed when $x = 5$. 3 MARKS

(c) A body of mass 4 kg is released from rest on a smooth plane inclined at 45°. Taking $g = 10$, find the normal reaction and the acceleration down the plane. 3 MARKS

(d) A body of mass m falls from rest against a resistance mkv. Show that $\dfrac{dv}{dt} = g - kv$. 2 MARKS

(e) Hence show that $v = \dfrac{g}{k}\left(1 - e^{-kt}\right)$. 2 MARKS

(f) State the terminal velocity when $g = 10$ and $k = 0.25$, and explain why it is never quite attained. 2 MARKS

End of paper.

About this resource

MA99 EDUCATION

Scope

NSW Mathematics Extension 2, Year 12. Written to the NSW Mathematics Extension 2 11-12 Syllabus (2024). For students sitting the HSC in 2027 and later. Five focus areas, matching the 2024 syllabus.

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Mathematics Extension 2: Practice Paper 1 · Worked Solutions

01

NSW MATHEMATICS EXTENSION 2 11-12 SYLLABUS 2024 FOR HSC 2027 AND LATER

Worked solutions · Section I

SECTION I KEY · 1 B · 2 D · 3 A · 4 C · 5 C · 6 B · 7 B · 8 A · 9 C · 10 D

1

The negation of "for every n, $P(n)$" is "there exists an n for which $P(n)$ is false".

Here $P(n)$ is "$n^2 + n$ is even", so its negation is "$n^2 + n$ is odd".

(B)

(A) negates the property but keeps the universal quantifier, which is a much stronger claim. (C) swaps the quantifier but forgets to negate the property. (D) negates the property twice and lands back on the original statement.

Negating a universal statement produces an existential one. Changing only the property, as (A) does, is the standard trap.

2

$n = 1$: 3, prime. $n = 2$: 7, prime. $n = 3$: 13, prime.

$n = 4$: $16 + 4 + 1 = 21 = 3 \times 7$, which is composite.

(D)

(A), (B) and (C) all give primes, so none of them is a counterexample.

A counterexample must actually falsify the claim. Test the options in order and stop at the first composite.

3

A point is on the line when one value of t reproduces all three components.

$(5, 0, 5)$: $1 + 2t = 5$ gives $t = 2$; then $2 - 2 = 0$ and $3 + 2 = 5$. All three agree.

(A)

(B) needs $t = 2$ from the first component but gives $y = 0$, not 4. (C) needs $t = 1$ and gives $y = 1$, not 3. (D) is the direction vector, not a point on the line.

Solve from one component, then CHECK the other two. Option (D) is the trap for anyone who confuses the direction with a point.

4

$(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} + \mathbf{k}) = 1 + 0 + 0 = 1$.

Both have magnitude $\sqrt{2}$, so $\cos\theta = \dfrac{1}{2}$.

(C) $60°$

(D) would need a zero scalar product. (B) would need $\cos\theta = \frac{1}{\sqrt{2}}$, which is what dividing by only one of the magnitudes gives. (A) is the angle whose SINE, not cosine, is $\frac{1}{2}$.

Divide by BOTH magnitudes. Dividing by only one gives 1 over root 2 and the wrong option (B).

5

$-1 - i\sqrt{3}$ is in the THIRD quadrant, with reference angle $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$.

A third quadrant principal argument is $\alpha - \pi = \frac{\pi}{3} - \pi$.

(C) $-\frac{2\pi}{3}$

(A) is the second quadrant answer. (B) names the same direction but is outside $(-\pi, \pi]$. (D) is the fourth quadrant answer.

Option (B) is the same point with a non principal argument. The interval is what rules it out.

6

$|1+i| = \sqrt{2}$ and $\operatorname{Arg}(1+i) = \frac{\pi}{4}$.

$(1+i)^6 = (\sqrt{2})^6 \operatorname{cis} \frac{6\pi}{4} = 8 \operatorname{cis} \frac{3\pi}{2} = 8 \operatorname{cis}\left(-\frac{\pi}{2}\right)$.

(B) $-8i$

(A) has the argument reduced the wrong way. (C) and (D) would need an argument of 0 or π.

Three pi over two reduces to minus pi over two, not plus. That sign is the whole difference between (A) and (B).

7

By parts with $u = x$ and $dv = e^x\,dx$: $xe^x - \int e^x\,dx$.

$= xe^x - e^x + C = (x-1)e^x + C$.

(B)

(C) has the sign of the second term wrong. (A) omits the second integration entirely. (D) treats e^x as a constant.

Differentiate the answer to check. The product rule on the chosen option must return the integrand, and only one option does. Two seconds, and it settles (B) against (C).

8

The numerator is the derivative of the denominator, so the integral is a logarithm.

$$\int \frac{2x}{x^2+1}\,dx = \ln(x^2+1) + C.$$

(A)

(B) and (C) misapply the coefficient. (D) is the integral of $\frac{2}{x^2+1}$, not of this.

Recognise f'(x) over f(x). The 2x is exactly the derivative of x squared plus 1, so no adjusting factor is needed.

9

$\ddot{x} = -n^2 x$ with $n^2 = 16$, so $n = 4$.

$$T = \frac{2\pi}{n} = \frac{2\pi}{4}.$$

(C) $\frac{\pi}{2}$

(B) uses $n = 8$. (D) computes $2\pi n$ instead of 2π over n. (A) uses $n = 16$, forgetting to take the square root.

Take the square root of 16 before dividing. Option (D) is the reciprocal error and is always offered.

10

Terminal velocity is where the acceleration vanishes: $10 - 0.2v = 0$.

$$v = \frac{10}{0.2}.$$

(D) 50

(A) multiplies where it should divide. (B) is $\sqrt{10/0.2}$, the answer for a SQUARE law resistance $10 - 0.2v^2$. (C) substitutes $v = 1$ instead of solving for v.

Set the acceleration to zero and solve. No integration is needed, which is why this is a one mark question.

Marking Section I

Ten marks sit in this section and they are the cheapest ten in the paper. Mark them honestly before reading a single Section II solution.

For every item you missed, name the ERROR, not the answer. Each distractor above is the answer that one specific mistake produces, and the note beside it names that mistake: that is the sentence to carry into the next paper.

An item you guessed correctly counts as an item you missed. Work it again from the stem with the options covered.

Worked solutions · Section II

11 (a)

Suppose $\sqrt{6}$ is rational, so $\sqrt{6} = \dfrac{p}{q}$ in lowest terms with $q \neq 0$.

Then $6q^2 = p^2$, so p^2 is even and hence p is even. Write $p = 2r$: $6q^2 = 4r^2$, so $3q^2 = 2r^2$.

Then $3q^2$ is even, and since 3 is odd, q^2 is even and so is q.

Both p and q are even, contradicting lowest terms, so $\sqrt{6}$ is irrational

The contradiction is with LOWEST TERMS, and that phrase must appear in the opening line or there is nothing to contradict.

11 (b)

Base case. $n = 1$: the left side is 1 and the right is $\frac{1(2)}{2} = 1$.

Hypothesis. Assume $1 + \cdots + k = \dfrac{k(k+1)}{2}$ for some positive integer k.

Step. $1 + \cdots + k + (k+1) = \dfrac{k(k+1)}{2} + (k+1) = (k+1)\left(\dfrac{k}{2} + 1\right) = \dfrac{(k+1)(k+2)}{2}$.

This is the statement for $k+1$, so by induction it holds for every positive integer n

Factor out (k+1) rather than expanding. The target form appears immediately and the algebra is one line.

11 (c)

$a + \dfrac{1}{a} - 2 = \dfrac{a^2 - 2a + 1}{a} = \dfrac{(a-1)^2}{a}$.

$(a-1)^2 \geq 0$ always, and $a > 0$ is given, so the quotient is non-negative.

$a + \dfrac{1}{a} \geq 2$

Subtract, put over a common denominator, factor. The condition a greater than zero is what makes the denominator safe.

11 (d)

Equality needs $(a-1)^2 = 0$.

$a = 1$

The equality case falls straight out of the factorisation in part (c). Do not start again.

11 (e)

$n^3 - n = n(n^2 - 1) = (n-1)n(n+1)$: three consecutive integers.

Among any three consecutive integers at least one is even, so the product is divisible by 2.

Among any three consecutive integers exactly one is divisible by 3, so the product is divisible by 3.

2 and 3 are coprime, so the product is divisible by 6

The coprime step is essential. Divisibility by 2 and by 3 gives divisibility by 6 only because 2 and 3 share no factor.

11 (f)

Take $n = 41$: $41^2 - 41 + 41 = 41^2 = 1681$.

$1681 = 41 \times 41$ is not prime, so $n = 41$ is a counterexample

Choosing n equal to 41 makes every term share a factor of 41. That is the structure, not luck.

11 (g)

A universal claim is about EVERY positive integer, and forty cases leave infinitely many untested.

Part (f) shows the very next case fails.

Verification is not proof. Part (f) is the demonstration that the distinction matters here.

12 (a)

$\vec{AB} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

Any point on the line and any non zero multiple of the direction earns full marks.

12 (b)

$7 = 1 + 2\lambda$ gives $\lambda = 3$.

Check: $1 + 3(3) = 10$ and $1 + 4(3) = 13$. Both agree.

C lies on the line, at $\lambda = 3$

One component gives a candidate; the other two confirm it. Stopping after the first is the standard loss.

12 (c)

Not parallel. $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ are not scalar multiples: $k = 2$ from the first component gives $2(-1) = -2 \neq 1$.

No intersection. Components: $1 + 2s = 3 + u$, $s = 2 - u$, $2 - s = 1 + u$.

From the second, $s = 2 - u$. Substituting into the first: $1 + 2(2 - u) = 3 + u$, so $5 - 2u = 3 + u$ and $u = \frac{2}{3}$, $s = \frac{4}{3}$.

Test the third: $2 - \frac{4}{3} = \frac{2}{3}$ but $1 + \frac{2}{3} = \frac{5}{3}$. It fails.

Neither parallel nor intersecting, so the lines are skew

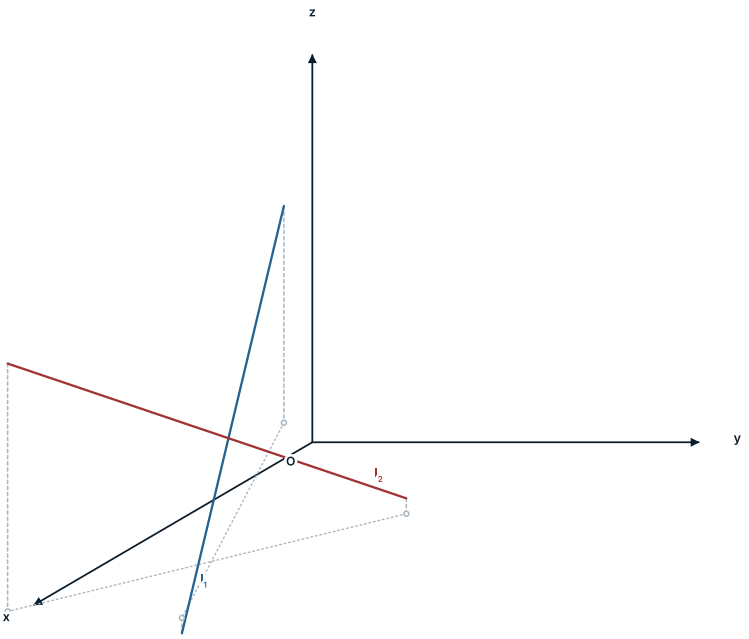


Figure P1.1 · The two lines of Question 12 (c). They are not parallel and the third component equation fails, so they are skew; the shortest distance between them is $\frac{1}{\sqrt{2}}$. On the page they may appear to cross, which the dashed drops to the xy plane dispel.

Both halves of the definition, and the concluding sentence naming them. That sentence is a mark in this course.

12 (d)

$$\mathbf{p}-\mathbf{c}=\begin{pmatrix}3\\3\\4\end{pmatrix}-\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}2\\1\\2\end{pmatrix}.$$

$|\mathbf{p}-\mathbf{c}|=\sqrt{4+1+4}=3$, and $3<6$.

P lies inside the sphere

Compute the distance to the CENTRE and compare with the radius. That is the whole method.

12 (e)

$$\begin{pmatrix}2\\2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix}=2-4+2=0.$$

Neither vector is zero.

They are perpendicular

The non zero remark matters: a zero vector has a zero scalar product with everything and no angle at all.

12 (f)

B has position vector $\mathbf{a}+\mathbf{c}$, so the diagonals are $\vec{OB}=\mathbf{a}+\mathbf{c}$ and $\vec{AC}=\mathbf{c}-\mathbf{a}$.

$(\mathbf{a}+\mathbf{c})\cdot(\mathbf{c}-\mathbf{a})=|\mathbf{c}|^2-|\mathbf{a}|^2$, the cross terms cancelling by commutativity.

This is zero exactly when $|\mathbf{a}|^2=|\mathbf{c}|^2$, and since both are non-negative, exactly when $|\mathbf{a}|=|\mathbf{c}|$.

The diagonals are perpendicular if and only if the adjacent sides are equal

Every step is reversible, which is what turns the chain into an if and only if. Say so.

13 (a)

Multiply above and below by $1+i$: the denominator becomes $1+1=2$.

Numerator: $(3+2i)(1+i)=3+3i+2i-2=1+5i$.

$\frac{1}{2} + \frac{5}{2}i$

Split into two fractions at the end. A single fraction over 2 is not the form a plus ib.

13 (b)

$|-2+2i| = \sqrt{4+4} = 2\sqrt{2}$.

Second quadrant, reference angle $\frac{\pi}{4}$, so $\text{Arg} = \pi - \frac{\pi}{4}$.

$|z| = 2\sqrt{2}, \text{Arg}(z) = \frac{3\pi}{4}$

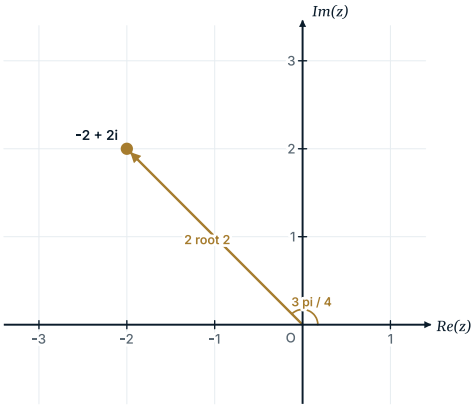


Figure P1.2 · $-2+2i$ lies in the second quadrant, so its principal argument is π MINUS the reference angle: $\frac{3\pi}{4}$, not $-\frac{\pi}{4}$.

Name the quadrant before converting. Second quadrant is pi minus the reference angle.

13 (c)

$|1+i\sqrt{3}| = 2$ and $\text{Arg} = \frac{\pi}{3}$.

By de Moivre, $\left(1+i\sqrt{3}\right)^6 = 2^6 \,\text{cis}\left(6 \times \frac{\pi}{3}\right) = 64\,\text{cis}(2\pi)$.

64

Two full turns, so the answer is a positive real. Reducing the angle before converting saves the arithmetic.

13 (d)

Let $(x+iy)^2 = -8+6i$ with x, y real: $x^2 - y^2 = -8$ and $2xy = 6$, so $xy = 3$.

$y = \frac{3}{x}$ gives $x^2 - \frac{9}{x^2} = -8$, so $x^4 + 8x^2 - 9 = 0$ and $(x^2+9)(x^2-1) = 0$.

x is real, so $x^2 = 1$ and $x = \pm 1$, with y of the same sign since $xy > 0$.

$1+3i$ and $-1-3i$

Reject x squared equals minus 9 with the words x is real. That rejection is a mark.

13 (e)

Let $z_1 = a+ib$ and $z_2 = c+id$ with a, b, c, d real.

$z_1 z_2 = (ac-bd) + i(ad+bc)$, so $\overline{z_1 z_2} = (ac-bd) - i(ad+bc)$.

$\bar{z}_1 \bar{z}_2 = (a-ib)(c-id) = (ac-bd) - i(ad+bc)$.

The two agree, so $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

Expand both sides independently and compare. Working from one to the other also earns the marks but is easier to get lost in.

13 (f)

$|z| = 1$ gives $z\bar{z} = 1$, so $\bar{z} = \dfrac{1}{z}$.

Let $w = \dfrac{1+z}{1-z}$. Then $\bar{w} = \dfrac{1+\bar{z}}{1-\bar{z}} = \dfrac{1+\frac{1}{z}}{1-\frac{1}{z}} = \dfrac{z+1}{z-1}$ after multiplying above and below by z.

$z - 1 = -(1 - z)$, so $\bar{w} = -w$, and $w \neq 0$ since $z \neq -1$.

$\dfrac{1+z}{1-z}$ is purely imaginary

Unit modulus turns the conjugate into a reciprocal. That single substitution is the whole proof.

13 (g)

Equality holds exactly when z_1 and z_2 have the same argument, or when one of them is 0.

Geometrically, the triangle collapses to a straight line.

The zero case is part of the condition and is the half most often omitted.

14 (a)

$16 = 16\,\mathrm{cis}(2k\pi)$, so $z = 2\,\mathrm{cis}\,\dfrac{2k\pi}{4} = 2\,\mathrm{cis}\,\dfrac{k\pi}{2}$ for $k = 0, 1, 2, 3$.

$2,\ 2i,\ -2,\ -2i$

They form a square of side $2\sqrt{2}$ on the circle of radius 2, and they sum to 0.

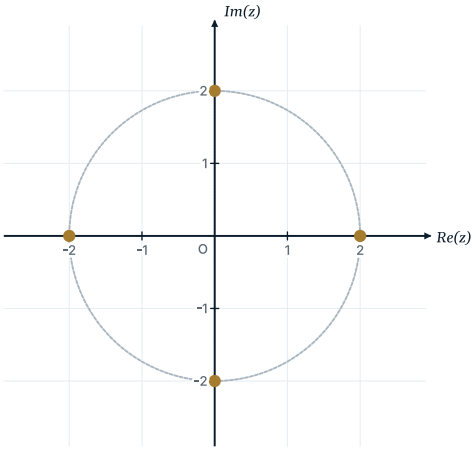


Figure P1.3 · The four solutions of $z^4 = 16$ sit on the circle of radius 2, a quarter turn apart, and sum to 0.

The fourth root of 16 is 2, not 4. The modulus takes the same root as the equation.

14 (b)

$-64i = 64\,\mathrm{cis}\left(-\frac{\pi}{2} + 2k\pi\right)$, and $64^{1/3} = 4$.

$z = 4\,\mathrm{cis}\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)$ for $k = 0, 1, 2$.

$k = 0$: $4\,\mathrm{cis}\left(-\frac{\pi}{6}\right) = 2\sqrt{3} - 2i$. $k = 1$: $4\,\mathrm{cis}\,\frac{\pi}{2} = 4i$. $k = 2$: $4\,\mathrm{cis}\,\frac{7\pi}{6} = 4\,\mathrm{cis}\left(-\frac{5\pi}{6}\right) = -2\sqrt{3} - 2i$.

$2\sqrt{3} - 2i,\ \ 4i,\ \ -2\sqrt{3} - 2i$

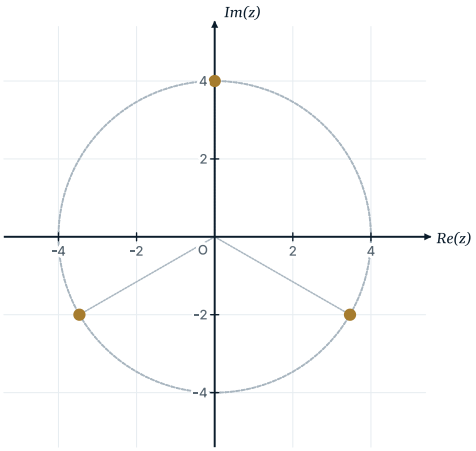


Figure P1.4 · The three solutions of $z^3 = -64i$ lie on the circle of radius 4, $\frac{2\pi}{3}$ apart. One of them, $4i$, is purely imaginary.

Their sum is zero, which checks all three at once. Reduce the third argument into the principal interval.

14 (c)

$|z - 2|$ is the distance to 2 and $|z + 2i| = |z - (-2i)|$ is the distance to $-2i$.

With $z = x + iy$: $(x-2)^2 + y^2 = x^2 + (y+2)^2$, so $-4x + 4 = 4y + 4$.

$y = -x$: the perpendicular bisector of the segment joining 2 and $-2i$

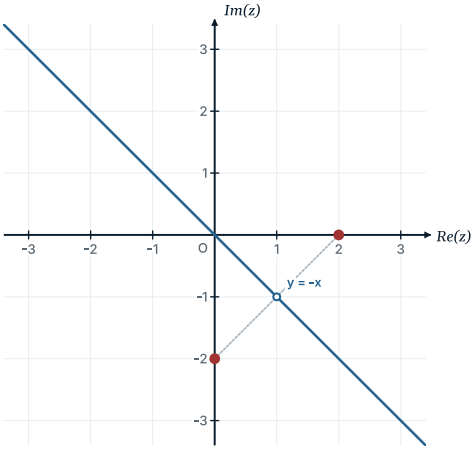


Figure P1.5 · The points equidistant from 2 and $-2i$ form the perpendicular bisector of the segment joining them, which is the line $y = -x$ through their midpoint $1 - i$.

Rewrite z plus 2i as z minus negative 2i before naming the point. The sign flips.

14 (d)

$\arg(z - 1) = \frac{\pi}{3}$ fixes the DIRECTION from the point 1 to z.

A ray from 1 at $\frac{\pi}{3}$ above the positive real direction, that is $y = \sqrt{3}(x - 1)$ with $x > 1$

The endpoint $z = 1$ is EXCLUDED, since $\arg 0$ is undefined.

A ray, not a line. The opposite direction has argument minus 2 pi over 3 and is not part of the locus.

14 (e)

$|z| \leq 3$is the closed disc of radius 3; $0 \leq \arg z \leq \frac{\pi}{4}$ is a wedge of angle $\frac{\pi}{4}$.

The intersection is a sector, of area $\frac{1}{2}r^2\theta = \frac{1}{2}(9)\left(\frac{\pi}{4}\right)$.

$\frac{9\pi}{8}$ square units

The origin is EXCLUDED, because $\arg 0$ is undefined. It is a single point, so the area is unaffected, but the exclusion still has to be said.

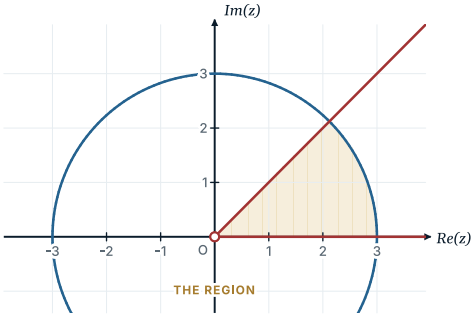


Figure P1.6 · $|z| \leq 3$AND $0 \leq \arg z \leq \frac{\pi}{4}$: a sector of angle $\frac{\pi}{4}$, whose area is $\frac{9\pi}{8} \approx 3.534$.

AND means intersection. Sketch each condition separately, then shade only the overlap. Any region defined by an argument condition loses its vertex.

14 (f)

The solutions of $z^n = c$ are the roots of $z^n - c = 0$.

For $n \geq 2$the coefficient of z^{n-1} in that polynomial is 0.

The sum of the roots is $-\dfrac{a_{n-1}}{a_n} = -\dfrac{0}{1}$.

The n roots sum to 0

Geometrically the roots are the vertices of a regular n-gon centred at the origin, which balances exactly.

15 (a)

The numerator is half the derivative of the denominator.

$$\int \frac{x}{x^2+4}\,dx = \tfrac{1}{2}\int \frac{2x}{x^2+4}\,dx.$$

$\frac{1}{2}\ln(x^2+4) + C$

Adjust by the constant factor rather than substituting. Here the factor is one half.

15 (b)

By parts with $u = x$, $dv = e^{2x}\,dx$: $\left[\tfrac{x}{2}e^{2x}\right]_0^1 - \tfrac{1}{2}\int_0^1 e^{2x}\,dx$.

$= \frac{e^2}{2} - \frac{1}{2}\left[\frac{e^{2x}}{2}\right]_0^1 = \frac{e^2}{2} - \frac{1}{4}\left(e^2 - 1\right)$.

$\frac{e^2}{4} + \frac{1}{4}$

The 1 over 2 from integrating e to the 2x appears twice. Losing one of them is the usual slip.

15 (c)

Apply parts to I_n with $u = x^n$ and $dv = e^{2x}\,dx$, so $v = \frac{1}{2}e^{2x}$.

$$I_n = \left[\tfrac{1}{2}x^n e^{2x}\right]_0^1 - \tfrac{n}{2}\int_0^1 x^{n-1}e^{2x}\,dx.$$

The boundary term is $\frac{1}{2}e^2 - 0$, and the remaining integral is I_{n-1}, so $I_n = \frac{1}{2}e^2 - \frac{n}{2}I_{n-1}$.

$2I_n = e^2 - nI_{n-1}$

The boundary term vanishes at the lower limit for every $n \geq 1$, which is why the relation starts there.

Differentiate the power and integrate the exponential, never the other way round: one application lowers the parameter by exactly one, and that is what manufactures the recurrence. Work towards the PRINTED form.

15 (d)

$n = 1$: $2I_1 = e^2 - I_0 = e^2 - \frac{1}{2}\left(e^2 - 1\right) = \frac{1}{2}\left(e^2 + 1\right)$, so $I_1 = \frac{1}{4}\left(e^2 + 1\right)$.

This agrees with part (b), which is I_1.

$n = 2$: $2I_2 = e^2 - 2I_1 = e^2 - \frac{1}{2}\left(e^2 + 1\right) = \frac{1}{2}\left(e^2 - 1\right)$.

$I_2 = \frac{1}{4}\left(e^2 - 1\right)$

Run the recurrence down to the base case rather than integrating again. Part (b) is I one, so it is a free check on the way through.

15 (e)

$\sin 3x \cos x = \frac{1}{2}\left[\sin 4x + \sin 2x\right]$.

$\int_0^{\pi/2} = \frac{1}{2}\left[-\frac{\cos 4x}{4} - \frac{\cos 2x}{2}\right]_0^{\pi/2}$.

At $\frac{\pi}{2}$: $\frac{1}{2}\left(-\frac{1}{4} + \frac{1}{2}\right) = \frac{1}{8}$. At 0: $\frac{1}{2}\left(-\frac{1}{4} - \frac{1}{2}\right) = -\frac{3}{8}$.

$\frac{1}{2}$

Both limits must be substituted. Cosines do not vanish at zero, so the lower limit contributes.

15 (f)

$m\ddot{x} = -mkv^2$, so $\ddot{x} = -kv^2$.

Distance is wanted, so use $v\dfrac{dv}{dx} = -kv^2$ and divide by v, which is non zero while the particle moves: $\dfrac{dv}{dx} = -kv$.

Separating, $\dfrac{dv}{v} = -k\,dx$, so $\ln v = -kx + C$, and $v = u$ at $x = 0$ gives $C = \ln u$.

$v = ue^{-kx}$

Dividing by v needs v non zero, and saying so is a mark. The speed decays with distance and never reaches zero.

16 (a)

$\dot{x} = 20\cos 4t$ and $\ddot{x} = -80\sin 4t$.

From the displacement, $\sin 4t = \dfrac{x-3}{5}$, so $\ddot{x} = -80\left(\dfrac{x-3}{5}\right) = -16(x-3)$.

This is $\ddot{x} = -n^2(x-c)$ with $n = 4$ and $c = 3$, so the motion is simple harmonic.

Centre $x = 3$, amplitude 5, period $\frac{2\pi}{4} = \frac{\pi}{2}$ seconds

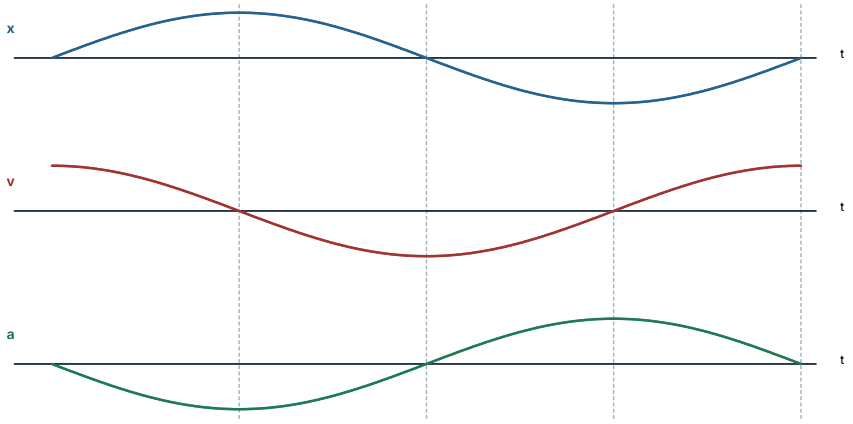


Figure P1.7 · $x = 5\sin 4t + 3$. The centre is 3, the amplitude 5 and the period $\frac{\pi}{2}$, all computed from the drawn motion. The maximum speed $An = 20$ occurs where the displacement crosses its centre.

Eliminate t between x and its second derivative. The centre is 3, not the origin.

16 (b)

$v_{\max} = An = 5 \times 4 = 20$ m/s, at the centre.

$$v^2 = n^2\left[A^2 - (x-c)^2\right] = 16\left[25 - (5-3)^2\right] = 16(21) = 336.$$

Maximum speed 20 m/s; speed at $x = 5$ is $\sqrt{336} = 4\sqrt{21} \approx 18.3$ m/s

Use the v squared equation rather than differentiating, because x is given and t is not.

16 (c)

The weight is $mg = 40$ N. Perpendicular to the plane there is no acceleration, so $N = 40\cos 45^\circ$.

Along the plane, $m\ddot{x} = mg\sin\theta$ and the mass cancels: $\ddot{x} = 10\sin 45^\circ$.

$N = 20\sqrt{2} \approx 28.3$ N; $\ddot{x} = 5\sqrt{2} \approx 7.07$ m/s^2

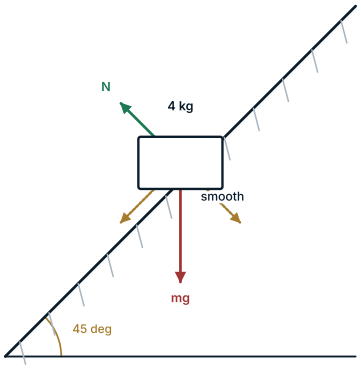


Figure P1.8 · At 45° the two components of the weight are EQUAL, both $20\sqrt{2}$ newtons. That equality is a free check on the arithmetic.

At 45 degrees the two components are equal, which is a useful check. The acceleration does not depend on the mass.

16 (d)

Take downwards as positive. Gravity acts downwards with magnitude mg; the resistance opposes the motion, so it acts upwards with magnitude mkv.

$m\ddot{x} = mg - mkv$, and dividing by m:

$$\frac{dv}{dt} = g - kv$$

Fix the positive direction first and say so. Every sign after that follows from the choice.

16 (e)

$\dfrac{dv}{g - kv} = dt$, so $-\frac{1}{k}\ln(g - kv) = t + C$.

At $t = 0, v = 0$: $C = -\frac{1}{k}\ln g$. Hence $\ln\dfrac{g - kv}{g} = -kt$ and $g - kv = ge^{-kt}$.

$$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

The minus one over k in the integration is where the sign errors live. Differentiate back to check.

16 (f)

$v_T = \dfrac{g}{k} = \dfrac{10}{0.25}$.

$v_T = 40$ m/s

From part (e), $v = 40\left(1 - e^{-t/4}\right)$, and $e^{-t/4} > 0$ for every finite t.

So the speed approaches 40 but never attains it

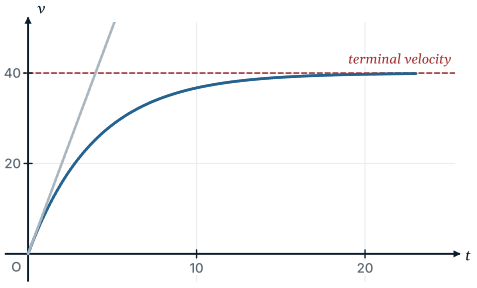


Figure P1.9 · $v = 40\left(1 - e^{-t/4}\right)$. The speed approaches 40 m/s and never attains it, because the exponential is positive for every finite t. The pale dashed line is the unresisted $v = gt$.

The word approaches is the mark. The exponential is positive for all finite t, which is the reason.

What this paper examined

Marks are counted per ITEM, not per question, so a mechanics derivation that happens to be carried out by integration is counted where it belongs. The target column is each focus area's share of the 100 marks, taken from its share of the Year 12 content points.

Focus area	Marks	Target
The nature of proof	17	21
Further work with vectors	17	16
Introduction to complex numbers	32	31
Further integration	14	11
Applications of calculus to mechanics	20	21
TOTAL	100	100

Using this paper

Work the whole paper in one sitting before reading a single solution. A worked solution read before the attempt teaches recognition, not method, and recognition is not what is examined.

Where your answer differs from the one printed, find the first line at which the two part company. That line, not the final answer, is what needs the work.

End of worked solutions.

About this resource

MA99 EDUCATION

Scope

NSW Mathematics Extension 2, Year 12. Written to the NSW Mathematics Extension 2 11-12 Syllabus (2024). For students sitting the HSC in 2027 and later. Five focus areas, matching the 2024 syllabus.

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