MA99 EDUCATION

Focus area 4 Further integration: Summary

04

NSW MATHEMATICS EXTENSION 2 11-12 SYLLABUS 2024 FOR HSC 2027 AND LATER

Further integration, on three pages

The standard forms

Integral	Result	Note
$\int x^n\,dx$	$\dfrac{x^{n+1}}{n+1}+C$, for $n\neq -1$	The exception is the log
$\int \dfrac{f'(x)}{f(x)}\,dx$	$\ln\lvert f(x)\rvert+C$	The pattern to look for first, every time
$\int \dfrac{dx}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}+C$	Sum of squares, no root
$\int \dfrac{dx}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert+C$	Difference of squares, no root
$\int \dfrac{dx}{\sqrt{a^2-x^2}}$	$\sin^{-1}\dfrac{x}{a}+C$	Difference under a root
$\int \dfrac{dx}{\sqrt{x^2+a^2}}$	$\ln\left(x+\sqrt{x^2+a^2}\right)+C$	Sum under a root
$\int \dfrac{dx}{\sqrt{x^2-a^2}}$	$\ln\left(x+\sqrt{x^2-a^2}\right)+C$	Difference under a root, x^2 first

The six methods, in one line each

Method	The formula	Chosen when
Substitution	$\int f(g(x))g'(x)\,dx=\int f(u)\,du$	One chunk differentiates to the rest
Inverse substitution	$x=a\sin\theta$ or $x=a\tan\theta$	A root or a power of $a^2\pm x^2$, with no derivative present
Partial fractions	One polynomial over another, decomposed	Always, for a rational function. Divide first if improper
By parts	$\int u\,dv=uv-\int v\,du$	A product in which one factor gets simpler
Reduction	I_n in terms of I_{n-1} or I_{n-2}	The integrand carries an integer n
The t formulae	$t=\tan\frac{x}{2}$	A rational function of $\sin x$ and $\cos x$, and nothing shorter works

The substitutions to find unprompted

- $\dfrac{f'}{f}$ present: write $\ln\lvert f\rvert$ down, no substitution needed.
- A bracket to a power with its derivative outside: $u=$ the bracket.
- A linear expression under a root: $u=$ that expression, then x in terms of u.
- $\sqrt{x}$ more than once: $u=\sqrt{x}$, so $x=u^2$ and $dx=2u\,du$.

- An odd power of $\sin x$: $u = \cos x$. An odd power of $\cos x$: $u = \sin x$.
- Even powers only: no substitution. Use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.
- $\sqrt{a^2 - x^2}$: $x = a\sin\theta$. $a^2 + x^2$ to a power: $x = a\tan\theta$.

Partial fractions · the form, by denominator

Denominator contains	Put above it	It integrates to
$(x - a)$	$\dfrac{A}{x - a}$	A logarithm
$(x - a)^2$	$\dfrac{A}{x - a} + \dfrac{B}{(x - a)^2}$	A logarithm **and** a power
$(x - a)^3$	Three terms, one over each power	A log and two powers
$x^2 + bx + c$, irreducible	$\dfrac{Bx + C}{x^2 + bx + c}$	A logarithm **and** an inverse tangent
Numerator degree at least the denominator's	Nothing yet: **divide first**	A polynomial plus the rest

Integration by parts · choosing u

Take u to be the first of these that is present: a logarithm, an inverse trigonometric function, a power of x. If the integrand is an exponential against a sine or cosine, either choice works and the integral loops, so name it I and solve.

The loop, written out

$I = \int e^{ax}\cos bx\,dx$. Two applications of parts in the same direction give $I = (\text{something}) - \dfrac{b^2}{a^2}I$, or with $a = b = 1$, $I = e^x\cos x + e^x\sin x - I$.

Then $2I = e^x(\sin x + \cos x)$ and $I = \frac{1}{2}e^x(\sin x + \cos x) + C$. The constant appears after the division, not before it.

Reduction formulae · the five steps

1. Define I_n, with its limits.
2. Split one factor off and name u, dv, du, v.
3. Apply parts and **write the boundary term out**. It does not always vanish.
4. Use a Pythagorean identity so that I_n reappears on the right.
5. Collect the I_n terms, divide, and state the range of n.

The t formulae

$$t = \tan\frac{x}{2}, \qquad \sin x = \frac{2t}{1 + t^2}, \qquad \cos x = \frac{1 - t^2}{1 + t^2}, \qquad dx = \frac{2\,dt}{1 + t^2}.$$

The $1 + t^2$ from dx cancels against the one from the integrand. If it does not, check your algebra before going further. Valid only where $\tan\frac{x}{2}$ is defined, so an interval containing π must be split.

Definite integral properties

- $\int_a^b f = -\int_b^a f$ and $\int_a^b f = \int_a^c f + \int_c^b f$.
- **Odd:** $f(-x) = -f(x)$ gives $\int_{-a}^{a} f = 0$.
- **Even:** $f(-x) = f(x)$ gives $\int_{-a}^{a} f = 2\int_0^a f$.

- **The king property:** $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$. Add the two forms and look for a constant integrand.
- **The consequence worth memorising:** $\int_0^a \dfrac{f(x)}{f(x)+f(a-x)}\,dx = \dfrac{a}{2}$.
- **And one more:** $\int_0^\pi x\,f(\sin x)\,dx = \dfrac{\pi}{2}\int_0^\pi f(\sin x)\,dx$.

The eight ways marks are lost here

1. Not stating the substitution, so the method mark has nothing to attach to.
2. Changing the limits and then substituting back to x as well.
3. Halving the wrong coefficient when completing the square on a negative x^2.
4. Omitting the lower power of a repeated partial fraction factor.
5. Putting a constant, not $Bx + C$, over an irreducible quadratic.
6. Assuming the boundary term in a reduction formula vanishes without checking.
7. Going round the parts loop a third time instead of solving for I.
8. Dropping the modulus in $\ln|f(x)|$ when f can be negative on the interval.

The reference sheet, and what to learn anyway

NESA prints a reference sheet with the examination. Before the examination, read the sheet published for your year and tick what it carries. Learn the four standard forms, the integral of $\frac{dx}{x^2-a^2}$, the substitution $dx = \frac{2\,dt}{1+t^2}$ and the derivation of each reduction formula in any case, so that the sheet is a check and not a crutch.

Where this sits

This document covers focus area 4 of the Year 12 course, Further integration, outcome ME2-12-04.

The full teaching for this material is Chapter 5, Further Integration of the MA99 Mathematics Extension 2 Year 12 Complete Textbook. This document is the layer between that textbook and the practice papers.

Mathematics Advanced and Mathematics Extension 1 are assumed knowledge for this course and for its examination.

Grounded in the NSW Mathematics Extension 2 11-12 Syllabus (2024), the syllabus in force for the first HSC examination in 2027.

About this resource

MA99 EDUCATION

Scope

NSW Mathematics Extension 2, Year 12. Written to the NSW Mathematics Extension 2 11-12 Syllabus (2024), focus area 4, Further integration (ME2-12-04), for students sitting the HSC in 2027 and later years. Every value, identity and integral was re-derived independently by computer algebra before printing.

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