2. Sequences and Series

Add the same amount each time and you get an arithmetic sequence; multiply by the same amount and you get a geometric one. Two patterns, a handful of formulas, and you can model a training plan, a falling tank, a growing town or a bouncing ball, and add up infinitely many numbers.

Year 12 Mathematics Advanced · written to the NSW Mathematics Advanced 11-12 Syllabus (2024), focus area Sequences and series.

What changed in 2024

Sequences and series is now a focus area of its own. Under the 2017 syllabus, arithmetic and geometric sequences were taught inside Year 12 Financial Mathematics, mainly as a tool for annuities and loans. NESA's engagement report on the draft 2024 syllabuses lists, among the strengths noted in feedback, “the separation of Sequences and series from Financial mathematics” (NESA, 2024).

The 2024 content adds formal definitions (a sequence, finite and infinite sequences, the nth partial sum, a series), **summation notation** and a proof by expansion of the identity behind the geometric sum formula.

The notation has changed: the nth term is a_n, not T_n, and the last term is written a_n, not l, so the sum formula reads $S_n = \frac{n}{2}(a + a_n)$. Recurrence relations as a topic of their own are Extension 2 work. Annuities and reducing balance loans use this chapter's formulas, but they belong to Chapter 7.

This chapter leans on two pieces of Year 11. An arithmetic sequence is a linear function in disguise (Year 11 Chapter 1), and a geometric sequence is an exponential function in disguise, so finding an unknown number of terms means solving an exponential equation or inequality with logarithms (Year 11 Chapter 6). If either feels rusty, revise it first.

2.1 Sequences and their terms

A **sequence** is a list of objects in a definite order. Each object is a **term**. Order matters and repeats are allowed: $1, 2, 3$ and $3, 2, 1$ are different sequences, and $1, 0, 1, 0, \ldots$ is a perfectly good one. In this course the objects are almost always numbers.

Sequences and the notation for terms

A sequence is an ordered list. Its terms are written $a_1, a_2, a_3, \ldots$, and a_n stands for the nth term, where n is a **positive integer**: $n = 1, 2, 3, \ldots$. There is no a_0 and no $a_{2.5}$.

A **finite** sequence $a_1, a_2, \ldots, a_m$ stops: its last term is the mth, for some whole number $m \geq 1$, so it has exactly m terms. An **infinite** sequence $a_1, a_2, a_3, \ldots$ never stops: every term has a next one.

A formula for a_n in terms of n, such as $a_n = n^2$, gives any term directly. Substitute the term number for n.

Sequence	Rule	Finite or infinite
$2, 4, 6, 8, \ldots$	$a_n = 2n$	infinite
$1, 4, 9, 16, \ldots$	$a_n = n^2$	infinite
$1, -1, 1, -1, \ldots$	$a_n = (-1)^{n+1}$	infinite
$1, 1, 2, 3, 5, 8, \ldots$	each term after the second is the sum of the two before it	infinite (the Fibonacci sequence)
$10, 20, 30, \ldots, 90$	$a_n = 10n$ for $1 \leq n \leq 9$	finite, $m = 9$ terms

The squares $1, 4, 9, 16, \ldots$ are neither of the two special kinds this chapter is about: the differences $3, 5, 7, \ldots$ are not constant and neither are the ratios. They are a good reminder that a sequence is just an ordered list with a rule. A sequence is also a function whose inputs are the term numbers: $a_n = f(n)$ with domain the positive integers (or $1, 2, \ldots, m$ for a finite sequence). Its graph is a set of separate points, one above each whole number, never a joined curve.

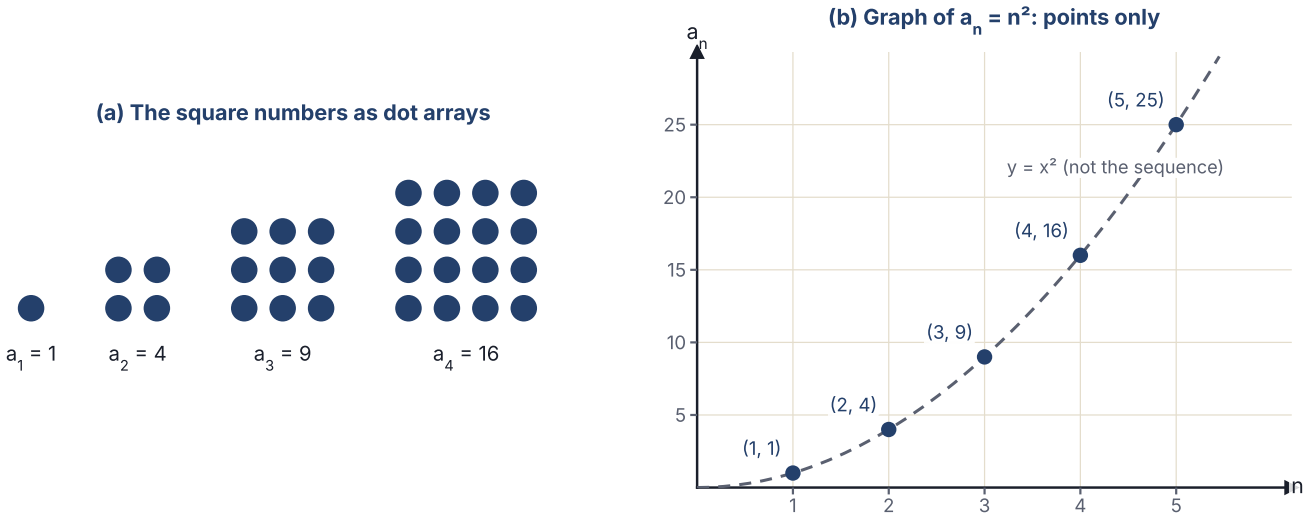


Figure 2.1 · (a) The square numbers $a_n = n^2$ count the dots in an n by n array. (b) Their graph is the set of points (n, n^2). The dashed curve $y = x^2$ passes through them, but only the points belong to the sequence.

Worked example 1

4 MARKS

The nth term of a sequence is $a_n = n^2 - 5n + 2$. (a) Find the first three terms. (b) Show that 86 is a term of the sequence and find its position. (c) Show that 30 is not a term.

01 (a) $a_1 = 1 - 5 + 2 = -2$, $a_2 = 4 - 10 + 2 = -4$, $a_3 = 9 - 15 + 2 = -4$: $-2, -4, -4$. repeats are allowed

02 (b) Solve $n^2 - 5n + 2 = 86$: $n^2 - 5n - 84 = 0$, so $(n - 12)(n + 7) = 0$ and $n = 12$ or $n = -7$.

03 n must be a positive integer, so $n = 12$: 86 is the 12th term, $a_{12} = 86$.
reject n = −7

04 (c) $n^2 - 5n + 2 = 30$ gives $n^2 - 5n - 28 = 0$, so $n = \dfrac{5 \pm \sqrt{137}}{2}$. $\sqrt{137}$ is not an integer, so neither root is a positive integer and 30 is not a term.

n is a counter.
Term numbers are 1, 2, 3, ... A negative or fractional n means 'not a term'.

Say why.
For (c), name the reason: the roots are not positive integers.

WHERE THE MARKS ARE

4 Correct solution to all parts. **3** Finds $n = 12$ with the negative root rejected, and makes progress in (c). **2** Correct terms and a correct quadratic for (b). **1** Correct first three terms. Leaving $n = -7$ standing beside $n = 12$ loses a mark.

Worked example 2

3 MARKS

Find a formula for the nth term of each sequence. (a) $3, 8, 15, 24, 35, \ldots$ (b) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ (c) $-1, 2, -3, 4, -5, \ldots$

01 (a) Each term is one less than a square: $4 - 1, 9 - 1, 16 - 1, \ldots$, the squares of $2, 3, 4, \ldots$ So $a_n = (n + 1)^2 - 1 = n^2 + 2n$. Check: $a_5 = 25 + 10 = 35$. compare with a sequence you know

02 (b) The numerator is the term number and the denominator is one more: $a_n = \dfrac{n}{n + 1}$.

Test it.
Substitute n = 1 and n = 5. A formula that fits the first term only is a guess.

Signs that alternate.
$(-1)^n$ starts negative, $(-1)^{n+1}$ starts positive.

03 (c) The size is n and the sign alternates, negative first. $(-1)^n$ is -1 when n is odd: $a_n = (-1)^n n$. $(-1)^n$ switches the sign

WHERE THE MARKS ARE

3 Three correct formulas. **2** Two correct. **1** One correct. A formula must reproduce every given term: check at least two of them.

A list of terms alone never forces one rule: $1, 2, 4, \ldots$ could continue $8, 16$ or $7, 11$. When a question gives only the first few terms, it wants the simplest rule that fits, and an answer should show it fits.

Worked example 3

2 MARKS

The two-digit multiples of 7 form a finite sequence $14, 21, 28, \ldots, 98$. Write a formula for a_n and find the number of terms m.

Count with the formula.
Set the formula equal to the last term and solve for m.

01 $14 = 7 \times 2, 21 = 7 \times 3$, so the nth term is 7 times $(n + 1)$: $a_n = 7n + 7$.

02 The last term: $7m + 7 = 98$, so $7m = 91$ and $m = 13$. 7 to 98 is 14 multiples; 7 itself is out

WHERE THE MARKS ARE

2 Correct formula and $m = 13$. **1** Correct formula, or a correct count with no formula. $\frac{98}{7} = 14$ is the usual slip: it counts the one-digit multiple 7 as well.

2.2 Partial sums and series

Adding the terms of a sequence as you go gives a second sequence, the running totals. These have their own name and their own notation.

Partial sums and series

The **nth partial sum** of a sequence $a_1, a_2, \ldots$ is the sum of its first n terms:

$$S_n = a_1 + a_2 + \cdots + a_n, \quad \text{for whole numbers } n \geq 1.$$

So $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, and the partial sums $S_1, S_2, S_3, \ldots$ form a sequence of their own.

A **series** is the sum of all the terms of an **infinite** sequence, written $a_1 + a_2 + a_3 + \cdots$ for the sequence $a_1, a_2, a_3, \ldots$ Whether a series adds up to a number is decided by what S_n does as n grows (section 2.9).

Each partial sum adds one term to the one before, $S_n = S_{n-1} + a_n$, so

$$a_n = S_n - S_{n-1} \text{ for } n \geq 2, \qquad a_1 = S_1.$$

Worked example 4

3 MARKS

The nth partial sum of a sequence is $S_n = 2n^2 + 3n$. (a) Find S_1, S_2 and S_3, and the first three terms. (b) Find a formula for a_n.

Sum, then subtract.
A partial sum is a running total. The newest term is the latest total minus the one before.

01 (a) $S_1 = 2 + 3 = 5$, $S_2 = 8 + 6 = 14$, $S_3 = 18 + 9 = 27$.

02 $a_1 = S_1 = 5, a_2 = S_2 - S_1 = 9, a_3 = S_3 - S_2 = 13$: $5, 9, 13$.

03 (b) For $n \geq 2$ $a_n = 2n^2 + 3n - \left[2(n-1)^2 + 3(n-1)\right]$ brackets round the whole of S(n−1)

04 $= 2n^2 + 3n - 2n^2 + 4n - 2 - 3n + 3 = \boxed{4n + 1}$.

05 At $n = 1$ the formula gives $5 = S_1$, so $a_n = 4n + 1$ holds for every $n \geq 1$.
always check n = 1 separately

Check n = 1.
S(n−1) means nothing at n = 1, so check a(1) against S(1) on its own.

WHERE THE MARKS ARE

3 Correct partial sums, terms and formula. **2** Uses $a_n = S_n - S_{n-1}$ with correct substitution. **1** Correct S_1, S_2, S_3. Writing $a_3 = S_3 = 27$ confuses a term with a sum and earns nothing for the terms.

Worked example 5

2 MARKS

Infinitely many terms.
A series can still have a finite total. Watch the partial sums.

For the series $\boxed{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots}$, find S_1, S_2, S_3 and S_4, and describe what happens to S_n as n increases.

01 $S_1 = \frac{1}{2}$, $S_2 = \frac{3}{4}$, $S_3 = \frac{7}{8}$, $S_4 = \frac{15}{16}$.

02 Each is 1 minus the last term added: $S_n = 1 - \left(\frac{1}{2}\right)^n$ fits all four. a pattern, proved in section 2.8

03 The gap $\left(\frac{1}{2}\right)^n$ halves every time, so $\boxed{S_n \text{ gets as close to 1 as we like}}$ but never reaches it.

WHERE THE MARKS ARE

2 Correct partial sums and a correct description. **1** Correct partial sums. Section 2.9 turns 'gets as close to 1 as we like' into the limiting sum $S_\infty = 1$.

Not every series settles down. For $1 - 1 + 1 - 1 + \cdots$ the partial sums are $1, 0, 1, 0, \ldots$, and for $1 + 2 + 3 + \cdots$ they are $1, 3, 6, 10, \ldots$, which grow past every number. Neither series adds up to a number.

2.3 Summation notation

Long sums are written compactly with the capital Greek letter sigma, Σ, which stands for 'sum'. NESA calls this **summation notation** (it is also called sigma notation).

Summation notation

For terms a_i to a_j of a sequence, with $j > i$:

$$\sum_{k=i}^{j} a_k = a_i + a_{i+1} + a_{i+2} + \cdots + a_{j-1} + a_j.$$

k is the **index**. It takes every whole number from the lower limit i to the upper limit j, and the terms are added. The index is a dummy letter: $\sum_{k=1}^{5} k^2$ and $\sum_{r=1}^{5} r^2$ are the same sum.

The sum has $j - i + 1$ terms. The nth partial sum is $S_n = \sum_{k=1}^{n} a_k$.

Because a finite sum can be added in any order: $\sum(a_k + b_k) = \sum a_k + \sum b_k$, $\sum c\,a_k = c\sum a_k$, and $\sum_{k=i}^{j} c = (j - i + 1)c$ for a constant c.

Worked example 6

2 MARKS

Write $\sum_{k=3}^{6}(k^2-2k)$ in full and evaluate it.

Write every term.
Four terms here. Listing them first makes the count visible.

01 $k=3,4,5,6$: $(9-6)+(16-8)+(25-10)+(36-12)$. 6 − 3 + 1 = 4 terms

02 $=3+8+15+24=\boxed{50}$.

WHERE THE MARKS ARE

2 Correct expansion and value. **1** Correct terms for at least three values of k. Starting at $k=1$, or stopping at $k=5$, are the usual errors.

Worked example 7

2 MARKS

Evaluate $\sum_{r=2}^{5}(3^r-4r)$.

The index is r.
Any letter can be the index. Substitute the values of the letter under the sigma.

01 $r=2$: $9-8=1$. $r=3$: $27-12=15$. $r=4$: $81-16=65$. $r=5$: $243-20=223$.

02 Sum: $1+15+65+223=\boxed{304}$.

03 Check by splitting: $\sum 3^r=9+27+81+243=360$ and $\sum 4r=4(2+3+4+5)=56$; $360-56=304$. the split is a free check

WHERE THE MARKS ARE

2 Correct answer. **1** Shows understanding of the notation, for example the correct four terms with one arithmetic slip. This is the style of NESA's sample question on summation notation.

Worked example 8

3 MARKS

Write each sum in summation notation. (a) $\boxed{5+9+13+\cdots+41}$ (b) $\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\frac{1}{81}+\frac{1}{243}$ (c) $2\times3+3\times4+4\times5+\cdots+20\times21$

Not unique.
Shift the index and the formula shifts with it. Both are correct.

Check both ends.
Substitute the lower and upper limits. You should get the first and last terms.

01 (a) The terms go up by 4 from 5, so $a_k=4k+1$. The last: $4k+1=41$ gives $k=10$. $\sum_{k=1}^{10}(4k+1)$. find the last index

02 (b) Size $\frac{1}{3^k}$, sign positive first: $\sum_{k=1}^{5}\frac{(-1)^{k+1}}{3^k}$.

03 (c) Each term is $k(k+1)$ with k from 2 to 20: $\sum_{k=2}^{20}k(k+1)$. the limits need not start at 1

WHERE THE MARKS ARE

3 Three correct. **2** Two correct. **1** One correct, or correct terms with wrong limits. Other forms are equally right: (a) is also $\sum_{k=0}^{9}(4k+5)$. Check the first and last terms of your answer against the question.

The trap · counting the terms

$\sum_{k=4}^{12} a_k$ has $12 - 4 + 1 = 9$ terms, not 8. Counting from i to j includes both ends. The same slip appears in Worked example 3 (14 multiples or 13?) and in every 'which term' question in this chapter. When unsure, count a tiny case: $\sum_{k=4}^{5}$ has two terms, 4 and 5.

Exercise 2A · Sequences, partial sums and summation notation

FOUNDATION One idea, applied once. The point of them is to check the idea is there.

1 Write the first four terms of the sequence $a_n = 3n - 7$. (1 mark)

2 Write the first four terms and the 10th term of $a_n = (-2)^n$. (2 marks)

3 Write $\sum_{k=1}^{4}(2k + 3)$ in full and evaluate it. (2 marks)

4 State whether each sequence is finite or infinite, and give the number of terms of any finite one. (a) $1, 3, 5, 7, \ldots$ (b) the three-digit multiples of 5, $100, 105, \ldots, 995$ (c) the digits of the decimal expansion of $\frac{1}{3}$ (2 marks)

STANDARD Two steps, or one step plus a decision. Most of the examination lives here.

5 $a_n = n^2 - 4n$. Find a_6, and the term number n for which $a_n = 45$. (2 marks)

6 Find a formula for the nth term. (a) $2, 5, 10, 17, 26, \ldots$ (b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ (c) $4, -8, 12, -16, \ldots$ (3 marks)

7 The nth partial sum of a sequence is $S_n = n^2 + 4n$. Find S_5, a_5, and a formula for a_n. (3 marks)

8 Evaluate $\sum_{k=2}^{6}(k^2 - 3k)$. (2 marks)

ADVANCED Three or more steps, or a method you must choose before you can start.

9 Write each sum in summation notation. (a) $4 + 7 + 10 + \cdots + 61$ (b) $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{100}$ (c) $-2 + 4 - 8 + 16 - \cdots - 512$ (3 marks)

10 Show that $\sum_{k=1}^{5}(k^2 + 3k) = \sum_{k=1}^{5} k^2 + 3\sum_{k=1}^{5} k$ by writing out each side, and evaluate it. (3 marks)

11 The nth partial sum of a sequence is $S_n = 3^n - 1$. Find a_1, a_2 and a_3, and show that $a_n = 2 \times 3^{n-1}$ for all $n \geq 1$. (3 marks)

12 How many terms are in $\sum_{k=7}^{40}(2k - 1)$? Rewrite the sum with an index that starts at 1. (3 marks)

EXAM-STYLE Multi-part, written the way the HSC writes them. Attempt these last.

13 A sequence has nth term $a_n = n^2 - 7n + 15$. (a) Find the first four terms. (b) Find the smallest term of the sequence. (c) Find which term is equal to 45. (d) Explain why no term is equal to 4. (5 marks)

14 The nth partial sum of a sequence is $S_n = 5n - n^2$. (a) Find a_1, a_2 and a_3. (b) Show that $a_n = 6 - 2n$. (c) Evaluate $\sum_{k=4}^{10} a_k$ in two ways: from the partial sums, and by adding the terms. (5 marks)

2.4 Arithmetic sequences

In an arithmetic sequence the pattern is made by **adding**: you get from each term to the next by adding the same number every time.

WORKING SPACE

Chapter 2 worked solutions

EXERCISE 2A

1

$a_1 = 3 - 7 = -4$, $a_2 = -1$, $a_3 = 2$, $a_4 = 5$: $-4, -1, 2, 5$.

Substitute $n = 1, 2, 3, 4$ in turn.

2

$-2, 4, -8, 16$, and $a_{10} = (-2)^{10} = 1024$.

An even power of a negative number is positive.

3

$(2 + 3) + (4 + 3) + (6 + 3) + (8 + 3) = 5 + 7 + 9 + 11 = 32$.

Four terms, one for each value of k from 1 to 4.

4

(a) Infinite: the odd numbers never stop.

(b) Finite: $a_n = 95 + 5n$, and $95 + 5m = 995$ gives $m = 180$ terms.

(c) Infinite: $\frac{1}{3} = 0.333\ldots$, and the digits $3, 3, 3, \ldots$ never stop.

A repeated term is still a term: (c) is an infinite sequence whose terms are all 3.

5

$a_6 = 36 - 24 = 12$.

$n^2 - 4n = 45$: $n^2 - 4n - 45 = (n - 9)(n + 5) = 0$, so $n = 9$ ($n = -5$ is not a term number).

Reject a negative root with a reason.

6

(a) One more than the squares: $a_n = n^2 + 1$.

(b) $a_n = \dfrac{1}{2^n}$.

(c) Size $4n$, positive first: $a_n = (-1)^{n+1} \times 4n$.

Check each formula at $n = 1$ and at one later term.

7

$S_5 = 25 + 20 = 45$ and $S_4 = 16 + 16 = 32$, so $a_5 = 45 - 32 = 13$.

$a_n = n^2 + 4n - \left[(n-1)^2 + 4(n-1)\right] = 2n - 1 + 4 = 2n + 3$ for $n \geq 2$ and $a_1 = 5 = S_1$, so it holds for all $n \geq 1$.

Check: $a_5 = 2 \times 5 + 3 = 13$, as found.

8

$k = 2, 3, 4, 5, 6$: $(4 - 6) + (9 - 9) + (16 - 12) + (25 - 15) + (36 - 18)$

$= -2 + 0 + 4 + 10 + 18 = 30$.

Five terms: $6 - 2 + 1 = 5$.

9

(a) $a_k = 3k + 1$ and $3k + 1 = 61$ at $k = 20$: $\sum_{k=1}^{20}(3k + 1)$.

(b) $\sum_{k=1}^{10} \frac{1}{k^2}$.

(c) $(-2)^k$ with $(-2)^9 = -512$: $\sum_{k=1}^{9}(-2)^k$.

Check the first and last terms of each answer against the question.

10

Left: $(1 + 3) + (4 + 6) + (9 + 9) + (16 + 12) + (25 + 15) = 4 + 10 + 18 + 28 + 40 = 100$.

Right: $(1 + 4 + 9 + 16 + 25) + 3(1 + 2 + 3 + 4 + 5) = 55 + 45 = 100$.

Both sides contain the same ten numbers, regrouped, so they are equal: 100.

This is why $\sum(a_k + b_k) = \sum a_k + \sum b_k$ and $\sum c\,a_k = c \sum a_k$.

11

$a_1 = S_1 = 2$; $a_2 = S_2 - S_1 = 8 - 2 = 6$; $a_3 = 26 - 8 = 18$: $2, 6, 18$.

For $n \geq 2$ $a_n = (3^n - 1) - (3^{n-1} - 1) = 3^n - 3^{n-1} = 3^{n-1}(3 - 1) = 2 \times 3^{n-1}$.

At $n = 1$: $2 \times 3^0 = 2 = S_1$. So $a_n = 2 \times 3^{n-1}$ for all $n \geq 1$.

Take out the common factor 3^{n-1}: $3^n = 3 \times 3^{n-1}$.

12

$40 - 7 + 1 =$ 34 terms.

Put $k = j + 6$, so j runs from 1 to 34 and $2k - 1 = 2j + 11$: $\sum_{j=1}^{34}(2j + 11)$.

Check: $j = 1$ gives $13 = 2 \times 7 - 1$, the first term of the original sum.

13

(a) $a_1 = 9, a_2 = 5, a_3 = 3, a_4 = 3$: $9, 5, 3, 3$.

(b) $a_n = \left(n - \frac{7}{2}\right)^2 + \frac{11}{4}$, least when n is closest to 3.5, that is $n = 3$ or 4: the smallest term is 3.

(c) $n^2 - 7n + 15 = 45$: $n^2 - 7n - 30 = (n - 10)(n + 3) = 0$, so $a_{10} = 45$.

(d) $n^2 - 7n + 15 = 4$ gives $n^2 - 7n + 11 = 0$, with $n = \frac{7 \pm \sqrt{5}}{2}$. Neither root is a whole number, so no term equals 4.

In (b) the vertex of the parabola is at $n = 3.5$, which is not a term number: test the whole numbers on either side.

14

(a) $S_1 = 4$, $S_2 = 6$, $S_3 = 6$, so $a_1 = 4, a_2 = 2, a_3 = 0$.

(b) $a_n = 5n - n^2 - \left[5(n-1) - (n-1)^2\right] = 5 - (2n - 1) = 6 - 2n$ for $n \geq 2$; $a_1 = 4 = S_1$, so $a_n = 6 - 2n$.

(c) From partial sums: $S_{10} - S_3 = (50 - 100) - 6 = -56$.

Adding: $-2 - 4 - 6 - 8 - 10 - 12 - 14 = -56$.

$\sum_{k=4}^{10} a_k$ leaves out the first three terms, so it is $S_{10} - S_3$, not $S_{10} - S_4$.