CHAPTER 3 · CALCULUS · MAV-12-04

3. Differential Calculus

Year 11 gave you the rules of differentiation and one special function, e^x, that is its own derivative. This chapter finishes the job: every exponential, logarithmic and trigonometric function in the course, the proofs of their derivatives, and the rules combined on any function you can build from them.

Year 12 Mathematics Advanced · written to the NSW Mathematics Advanced 11-12 Syllabus (2024), focus area Differential calculus (MAV-12-04).

The chapter follows NESA's four content groups in order. Sections 3.1 to 3.3 differentiate exponential functions, 3.4 to 3.6 logarithmic functions and 3.7 to 3.9 trigonometric functions. Sections 3.10 and 3.11 put the product, quotient and chain rules to work on combinations of all of them, and finish with tangents and normals.

What the 2024 syllabus asks of this chapter

Proofs are content. You prove the derivative of e^{ax+b} with the chain rule, the derivative of a^x from $a = e^{\ln a}$, the derivative of $\ln x$ from the identity $e^{\ln x} = x$, and the derivative of $\log_a x$ from the change of base rule. Expect show that and prove questions on each.

Radians. The derivatives of $\sin x$ and $\cos x$ are found from their graphs and checked with a graphing application, and they hold only when x is measured in radians. Section 3.7 shows why.

New in 2024. You differentiate $\operatorname{cosec} x$, $\sec x$ and $\cot x$ yourself with the rules (NESA says the results need not be memorised), the proof of $\frac{d}{dx}(\ln x)$ starts from $e^{\ln x} = x$, and ke^{ax}, a Year 11 point in the 2017 course, now sits here.

Not in this chapter. Second derivatives, the nature of stationary points, curve sketching, optimisation and the growth and decay model $\frac{dQ}{dt} = kQ$ are Chapter 5. Integration is Chapter 4.

What you need from Year 11

The power, product, quotient and chain rules, and tangents and normals (Year 11 Chapter 5). The facts $\frac{d}{dx}(e^x) = e^x$, $\log_a a^x = x = a^{\log_a x}$, the log laws and the change of base rule (Year 11 Chapter 6). Radian measure, exact values, $\sec x$, $\operatorname{cosec} x$, $\cot x$ and the Pythagorean identities (Year 11 Chapters 3 and 4). If any of these is shaky, go back first: this chapter uses them on every page.

3.1 The derivatives of ke^{ax} and e^{ax+b}

Year 11 established that e is the one base whose exponential is its own derivative: $\frac{d}{dx}(e^x) = e^x$. At every point of $y = e^x$ the gradient of the tangent equals the height. Everything in this section is that fact combined with the chain rule.

Take $y = e^{ax+b}$, for constants $a \neq 0$ and b. The index is a linear function of x, so let $u = ax + b$. Then $y = e^u$, and

$$\frac{dy}{du} = e^u \quad \text{and} \quad \frac{du}{dx} = a \qquad e^u \text{ is its own derivative}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times a \qquad \text{the chain rule}$$

$$= ae^{ax+b} \qquad \text{replace } u \text{ by } ax + b$$

That is the proof NESA asks for. Setting $b = 0$ gives $\frac{d}{dx}(e^{ax}) = ae^{ax}$, and a constant factor k waits outside, by the constant multiple rule.

Exponentials with a linear index

$$\frac{d}{dx}\left(e^{ax+b}\right) = ae^{ax+b} \qquad \frac{d}{dx}\left(ke^{ax}\right) = ake^{ax}$$

where a, b and k are constants. In words: the derivative is the same exponential, multiplied by the coefficient of x in the index.

Worked example 1

3 MARKS

Differentiate (a) $y = 5e^{3x}$ (b) $y = e^{2-4x}$ (c) $y = \dfrac{3}{e^{2x}}$.

01 (a) $k = 5$, $a = 3$: $\dfrac{dy}{dx} = 3 \times 5e^{3x} = 15e^{3x}$.

02 (b) The coefficient of x in the index is -4: $\dfrac{dy}{dx} = -4e^{2-4x}$. the sign comes down too

03 (c) Rewrite first: $y = 3e^{-2x}$, so $\dfrac{dy}{dx} = -2 \times 3e^{-2x} = -6e^{-2x}$, which is $-\dfrac{6}{e^{2x}}$. a reciprocal is a negative index

Same exponential.
The index never changes when you differentiate e to a linear index.

Coefficient out front.
Multiply by the number in front of x in the index, sign included.

WHERE THE MARKS ARE

3 · all three derivatives correct. **2** · two correct. **1** · one correct, or the right exponential in each part with a wrong multiplier. In (c) the quotient rule also works, but rewriting with a negative index is one line.

Worked example 2

3 MARKS

Let $f(x) = 4e^{0.5x} - 6x$. (a) Find the gradient of the tangent at $x = 0$. (b) Find the exact value of x at which the tangent is horizontal.

Undo e with ln.
If e to the power of something equals 3, that something is ln 3.

01 $f'(x) = 0.5 \times 4e^{0.5x} - 6 = 2e^{0.5x} - 6$.

02 (a) $f'(0) = 2e^0 - 6 = 2 - 6 = -4$. e to the 0 is 1

03 (b) Horizontal tangent: $2e^{0.5x} - 6 = 0$, so $e^{0.5x} = 3$.

04 Take natural logarithms: $0.5x = \ln 3$, so $x = 2\ln 3$. exact means leave ln 3

WHERE THE MARKS ARE

1 · correct $f'(x)$. **1** · $f'(0) = -4$. **1** · $x = 2\ln 3$ (or $\ln 9$). A decimal such as 2.197 does not earn the last mark when the question says exact.

Some exponential expressions are easier after a little algebra. A product or quotient of exponentials collapses into a sum by the index laws, and then each term is a linear index.

Worked example 3

3 MARKS

Differentiate (a) $y = (e^x + 2)^2$, in two ways (b) $y = \dfrac{e^{2x} + 1}{e^x}$.

Two routes, one answer.
Expanding is a check on the chain rule, not a replacement for it.

Split the fraction.
Each term over e to the x, then subtract indices.

01 (a) Chain rule: $\dfrac{dy}{dx} = 2(e^x + 2) \times e^x = 2e^x(e^x + 2)$. the inside derivative is e to the x

02 Expanding: $y = e^{2x} + 4e^x + 4$, so $\dfrac{dy}{dx} = 2e^{2x} + 4e^x$, the same.

03 (b) Divide each term: $y = e^{2x-x} + e^{-x} = e^x + e^{-x}$. index law

04 $\dfrac{dy}{dx} = e^x - e^{-x}$.

WHERE THE MARKS ARE

2 · part (a) by either method. **1** · part (b). Note $(e^x)^2 = e^{2x}$, not e^{x^2}: squaring doubles the index.

The trap: using the power rule on an exponential

The power rule needs the variable in the base and a constant index. In e^{3x} the base is a constant and the variable is in the index, so $\frac{d}{dx}(e^{3x}) = 3xe^{3x-1}$ is wrong. The correct derivative is $3e^{3x}$: the index never drops.

3.2 The derivative of a^x

Year 11 Chapter 6 showed, from first principles, that $\frac{d}{dx}(a^x) = ka^x$, where k is the gradient of $y = a^x$ at its y intercept, and it found k only numerically: about 0.693 for $a = 2$ and about 1.099 for $a = 3$. Now k can be found exactly. The key is to write any base as a power of e.

For $a > 0$, the inverse property $e^{\ln a} = a$ gives $a^x = \left(e^{\ln a}\right)^x = e^{x\ln a}$. This is on the reference sheet as $a^x = e^{x\ln a}$. Since $\ln a$ is a constant, the index $x\ln a$ is linear in x, and Section 3.1 applies:

$a^x = e^{x \ln a}$ $a = e^{\ln a}$ and the index law $(e^m)^x = e^{mx}$

$\dfrac{d}{dx}(a^x) = \dfrac{d}{dx}\left(e^{(\ln a)x}\right) = (\ln a)\, e^{x \ln a}$ $\frac{d}{dx}(e^{mx}) = me^{mx}$ with $m = \ln a$

$= (\ln a)\, a^x$ $e^{x \ln a} = a^x$ again

Exponentials with any base

$$\frac{d}{dx}(a^x) = (\ln a)\, a^x, \qquad a > 0$$

It is on the reference sheet, but the proof is not, and proving it is a content point. The Year 11 constant k is $\ln a$: $\ln 2 = 0.693\ldots$, $\ln 3 = 1.0986\ldots$, $\ln 10 = 2.302\ldots$, and $\ln e = 1$, which is why e^x is its own derivative.

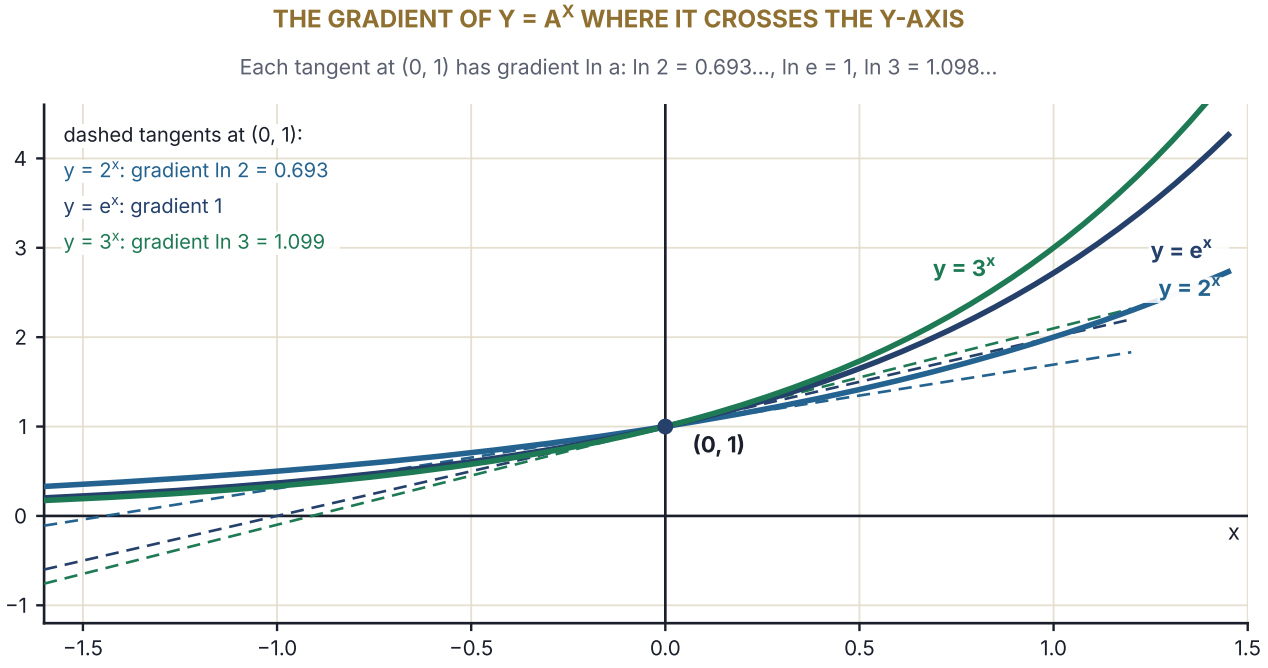


Figure 3.1 · The graphs of $y = 2^x$, $y = e^x$ and $y = 3^x$ all pass through $(0, 1)$. The gradient there is $\ln a$, so the curve with $a = e$ is the one whose tangent at $(0, 1)$ has gradient exactly 1.

Worked example 4

3 MARKS

Differentiate (a) $y = 5^x$ (b) $y = 3 \times 2^x$ (c) $y = 10^{-x}$.

ln a, not ln x.
The multiplier is a number: ln 5 is about 1.609.

01 (a) $\dfrac{dy}{dx} = (\ln 5)\, 5^x$.

02 (b) The constant waits outside: $\dfrac{dy}{dx} = 3(\ln 2)\, 2^x$.

03 (c) $y = 10^{-x} = e^{-x \ln 10}$, so $\dfrac{dy}{dx} = -\ln 10 \times e^{-x \ln 10} = -(\ln 10)\, 10^{-x}$. rewrite as a power of e when the index is not just x

WHERE THE MARKS ARE

3 · all three correct. **2** · two correct. **1** · one correct. In (c) the minus sign comes from the index, exactly as in Section 3.1; writing $10^{-x} = \left(\frac{1}{10}\right)^x$ and using $\ln \frac{1}{10} = -\ln 10$ gives the same answer.

Worked example 5

3 MARKS

(a) Find the exact gradient of the tangent to $y = 2^x$ at $x = 3$. (b) Find, correct to two decimal places, the value of x at which the gradient of the tangent is 1.

01 $\dfrac{dy}{dx} = (\ln 2)\, 2^x$.

02 (a) At $x = 3$: $(\ln 2) \times 8 = 8 \ln 2$.

03 (b) $(\ln 2)\, 2^x = 1$, so $2^x = \dfrac{1}{\ln 2}$, and $x = \log_2\left(\dfrac{1}{\ln 2}\right) = -\dfrac{\ln(\ln 2)}{\ln 2}$. change of base to evaluate

04 $x = -\dfrac{-0.3665\ldots}{0.6931\ldots} = 0.5287\ldots$, so $x \approx 0.53$.

Exact keeps ln.
8 ln 2 is exact. 5.545 is not.

Isolate the power.
Get 2 to the x alone before taking logs.

WHERE THE MARKS ARE

1 · correct derivative. **1** · $8 \ln 2$. **1** · $x \approx 0.53$. The answer to (b) is positive, so a negative value signals a lost minus sign: $\ln(\ln 2)$ is negative because $\ln 2 < 1$.

Beyond the syllabus (not essential content)

The constant $\ln a$ is the limit $\lim\limits_{h \to 0} \dfrac{a^h - 1}{h}$ from the first principles argument of Year 11. Evaluating it with technology for several values of a, and comparing it with $\ln a$, is an extension NESA lists as not essential. You will not be asked to do it.

The trap: power rule again, or ln x for ln a

$\frac{d}{dx}(2^x)$ is not $x \cdot 2^{x-1}$, for the same reason as in Section 3.1: the variable is in the index. And the multiplier is $\ln a$, not $\ln x$: $\frac{d}{dx}(5^x) = (\ln 5)5^x$, and $\ln 5$ is just a number.

3.3 The chain rule with $e^{f(x)}$

When the index is any differentiable function $f(x)$, the same chain rule argument works with $u = f(x)$: $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = f'(x)$, so the derivative is the same exponential multiplied by the derivative of the index.

Exponential of a function

$$\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)\, e^{f(x)}$$

This is on the reference sheet. Section 3.1 is the special case $f(x) = ax + b$, where $f'(x) = a$.

Worked example 6

3 MARKS

Differentiate (a) $y = e^{x^2}$ (b) $y = e^{\sqrt{x}}$ (c) $y = e^{\frac{1}{x}}$.

01 (a) $f(x) = x^2$, $f'(x) = 2x$: $\dfrac{dy}{dx} = 2xe^{x^2}$.

Name the index.
Write f(x) and f dash of x on one line. Then assemble.

Index unchanged.

02 (b) $f(x) = x^{\frac{1}{2}}$, $f'(x) = \dfrac{1}{2\sqrt{x}}$: $\dfrac{dy}{dx} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$.

03 (c) $f(x) = x^{-1}$, $f'(x) = -\dfrac{1}{x^2}$: $\dfrac{dy}{dx} = -\dfrac{e^{\frac{1}{x}}}{x^2}$.

The exponential is copied exactly. Only a multiplier is added.

WHERE THE MARKS ARE

3 · all three correct. **2** · two correct. **1** · one correct, or the exponential unchanged in each part with a wrong or missing $f'(x)$.

Worked example 7

3 MARKS

For the curve $y = e^{x^2-2x}$: (a) find the point where the tangent is horizontal; (b) find the gradient of the tangent at $x = 2$.

Exponentials are positive.
In f dash times e to the f, only f dash can be zero.

01 $\dfrac{dy}{dx} = (2x-2)e^{x^2-2x}$.

02 (a) $e^{x^2-2x} > 0$ for every x, so $\dfrac{dy}{dx} = 0$ only when $2x - 2 = 0$, that is $x = 1$. an exponential is never zero

03 $y = e^{1-2} = e^{-1}$, so the point is $\left(1, \dfrac{1}{e}\right)$.

04 (b) At $x = 2$: $(4-2)e^{4-4} = 2e^0 = 2$.

WHERE THE MARKS ARE

1 · correct derivative. **1** · the point $\left(1, e^{-1}\right)$, with the reason that the exponential factor cannot be zero. **1** · gradient 2.

The same idea differentiates $a^{f(x)}$: rewrite it as $e^{f(x)\ln a}$. For example, $3^{x^2} = e^{x^2 \ln 3}$, so $\frac{d}{dx}\left(3^{x^2}\right) = 2x \ln 3 \times e^{x^2 \ln 3} = 2x(\ln 3)\, 3^{x^2}$.

The trap: leaving out f dash of x

The derivative of the index must be there. $\frac{d}{dx}\left(e^{x^2}\right)$ is not e^{x^2}, and it is not $x^2 e^{x^2-1}$. Only e^x itself is its own derivative; every other index brings its own derivative down as a factor.

Exercise 3A · Differentiation with exponential functions

FOUNDATION One idea, applied once. The point of them is to check the idea is there.

1 Differentiate (a) $y = e^{4x}$ (b) $y = 7e^x$ (c) $y = 2e^{-3x}$ (d) $y = e^{5x+2}$. (2 marks)

2 Differentiate (a) $y = 6^x$ (b) $y = 4 \times 3^x$ (c) $y = \left(\frac{1}{2}\right)^x$. (2 marks)

3 Differentiate (a) $y = e^{x^3}$ (b) $y = e^{2x^2-x}$. (2 marks)

STANDARD Two steps, or one step plus a decision. Most of the examination lives here.

4 Write $y = \dfrac{e^{2x} - 3}{e^x}$ as a sum of exponentials, and hence differentiate it. (2 marks)

5 Find the exact gradient of the tangent to $y = 3e^{2x-1}$ at $x = \frac{1}{2}$ and at $x = 0$. (2 marks)

6 Differentiate (a) $y = (e^x + 1)^3$ (b) $y = \sqrt{e^{2x} + 5}$. (3 marks)