MA99 EDUCATION

Focus area 1 Proof by mathematical induction: Summary

01

NSW MATHEMATICS EXTENSION 1 11-12 SYLLABUS 2024 FOR HSC 2027 AND LATER

The structure

To prove $P(n)$ for every integer $n \geq a$:

Part	What you write
Statement	Let $P(n)$ be the statement ... for integers $n \geq a$(use $S(n)$ if a polynomial P is about)
Base case	Evaluate both sides at $n = a$ and show they agree (or show divisibility). $P(a)$ is true.
Assumption	Assume $P(k)$ is true for **some** integer $k \geq a$, written out in full.
Target	Prove $P(k+1)$, written out with every n replaced by $k+1$.
Inductive step	From the LHS of $P(k+1)$, use the assumption (say where), reach the RHS.
Conclusion	$P(a)$ is true, and $P(k)$ true implies $P(k+1)$ true, so by mathematical induction $P(n)$ is true for all integers $n \geq a$

Why it works: the base case starts the chain and the step passes truth from each integer to the next, like a line of dominoes. Checking cases, however many, is not a proof.

Sums

$$\sum_{r=1}^{k+1} T_r = \underbrace{\sum_{r=1}^{k} T_r}_{\text{use } P(k)} + T_{k+1}$$

1. Base case: write out the terms; $1 + 2 + \cdots + 2^n$ has one term at $n = 0$ and two at $n = 1$.
2. Step: LHS of $P(k+1)$ = RHS of $P(k)$ + T_{k+1}.
3. Take out the common factor, for example $(k+1)(k+2)$, instead of expanding.
4. Fractions: one common denominator; the target says which factor cancels.
5. Factorials: $(k+2)! = (k+2)(k+1)!$.

Worked results in the notes	Step in one line
$\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$	$\frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$
$1 + 2 + \cdots + 2^n = 2^{n+1} - 1, n \geq 0$	$2^{k+1} - 1 + 2^{k+1} = 2^{k+2} - 1$
$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$	$\frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$
$\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$	$1 - \frac{k+2}{(k+2)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$
$\sum_{r=1}^{n} (-1)^{r+1} r^2 = (-1)^{n+1} \frac{n(n+1)}{2}$	take out $(-1)^{k+2}(k+1)$: $-\frac{k}{2} + (k+1) = \frac{k+2}{2}$
$\sum_{r=2}^{n} \frac{1}{r^2-1} = \frac{(n-1)(3n+2)}{4n(n+1)}, n \geq 2$	base case $n = 2$; numerator $3k^3 + 5k^2 = k^2(3k+5)$

Hence after a proof means use the formula: for example $1^2 - 2^2 + \cdots - 20^2$ is the case $n = 20$, giving -210; a sum from $r = 11$ to 20 is (sum to 20) minus (sum to 10).

Divisibility

A is divisible by d when $A = dM$ for an **integer** M. Assume $f(k) = dM$; show $f(k+1) = d\times$ (an integer), and say it is an integer.

Move	Example
Substitute: make the power the subject	$3^{2k} = 8M - 7$, so $9 \times 3^{2k} + 7 = 8(9M - 7)$
Add and subtract	$6^{k+1} - 1 = 6(6^k - 1) + 5$
Two bases: replace one	$7(4M + 3^k) - 3 \times 3^k = 4(7M + 3^k)$
Polynomial: expand, find $f(k)$	$(k+1)^3 + 5(k+1) = (k^3 + 5k) + 3k(k+1) + 6$; $k(k+1)$ is even
Odd n only	base $n = 1$, step $k \to k+2$: $4 \times 2^k + 9 \times 3^k$

The algebra moves in the step

Piece of $P(k+1)$	Rewrite as
a^{k+1}; $a^{2(k+1)}$; $a^{2(k+1)-1}$	$a \times a^k$; $a^2 \times a^{2k}$; $a^2 \times a^{2k-1}$
$(k+2)!$	$(k+2)(k+1)!$
$(-1)^{k+2}$	$-(-1)^{k+1}$
$(k+1)^3$	$k^3 + 3k^2 + 3k + 1$
$(k+1)^2 - 1$	$k(k+2)$
$k(k+1)$	$2N$, as one of two consecutive integers is even

A full mark proof, in miniature

Let $P(n)$ be … . **Base:** $n = 1$, LHS $= \ldots$, RHS $= \ldots$, so $P(1)$ is true. **Assume** $P(k)$ for some integer $k \geq 1$: (written out). **Prove** $P(k+1)$: (written out). **LHS** $= \ldots$ (by the assumption) $= \ldots =$ RHS. **So** $P(k) \Rightarrow P(k+1)$; as $P(1)$ is true, by mathematical induction $P(n)$ is true for all integers $n \geq 1$

Marking, as on NESA's sample paper for a 3 mark proof: **1** mark for the base case, **2** for the base case with a sound inductive step, **3** for the complete, correct proof.

False proofs

What fails	Example	How to show it
Base case (step holds)	$2 + 4 + \cdots + 2n = n^2 + n + 1$; $4^n + 1$ divisible by 3	do the step, then compute $P(1)$ (or show it fails for all n)
Step (base holds)	$n^2 - n + 41$ prime: true to $n = 40$, false at 41	a counterexample
A line of the step	$3(3^k + 1)$ written as $3^{k+1} + 1$	expand and correct the line
Step needs larger k	birth month "proof": overlap of $k - 1$ is empty at $k = 1$	test the step at k equal to the base value

The traps, and can I do it

1. Assuming $P(n)$ for all n, or $P(k+1)$, instead of $P(k)$ for some k.
2. A base case asserted, not computed, or with the wrong number of terms.
3. Expanding instead of taking out the common factor.
4. Not saying the bracket in a divisibility proof is an integer.
5. A step to $k+1$ when the statement is for odd n.
6. Leaving off the conclusion, or giving the wrong values of n.
7. Proving a result another way when the question says by induction.

Tick each one you can do without notes. The number is the section of the Topic Notes.

- ☐ Explain the base case and the inductive step, and why both are needed (1)
- ☐ Lay out a proof with statement, base case, assumption, target, step and conclusion (1)
- ☐ Prove a sum with a product, a power of 2, a fraction or a factorial in it (2)
- ☐ Prove a divisibility result by substituting, or by adding and subtracting (3)
- ☐ Prove a divisibility result for a polynomial in n, and for odd n only (3)
- ☐ Show a step holds for a false statement, and show the statement fails (4)
- ☐ Find the false line, or the failing value of k, in a flawed proof (4)

Written to the NSW Mathematics Extension 1 11-12 Syllabus (2024), the syllabus Year 12 studies from Term 4 2026, first examined in the 2027 HSC. Year 11 work is assumed, and HSC questions can draw on it.

About this resource

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Scope

NSW Mathematics Extension 1, Year 12. Written to the NSW Mathematics Extension 1 11-12 Syllabus (2024), focus area 1, Proof by mathematical induction (ME1-12-01), for students sitting the HSC in 2027 and later years. Every value, identity and solution was re-derived independently by computer algebra before printing. Contexts in the questions are invented unless a source is named.

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