MA99 EDUCATION

Focus area 1 Further work with functions: Summary

01

NSW MATHEMATICS EXTENSION 1 11-12 SYLLABUS 2024 FOR HSC 2028 AND LATER

Graphical relationships

The reciprocal $y = \dfrac{1}{f(x)}$

On the original graph	On the reciprocal graph
zero at $x = a$	vertical asymptote $x = a$
$f \to \pm\infty$ (vertical asymptote, or as $x \to \pm\infty$)	$\frac{1}{f} \to 0$
horizontal asymptote $y = c, c \neq 0$	horizontal asymptote $y = \frac{1}{c}$
$f = 1$ or $f = -1$	same point: the graphs meet
sign of f	same sign
increasing, local minimum (a, b)	decreasing, local maximum $\left(a, \frac{1}{b}\right)$

The reciprocal trigonometric functions

	Secant	Cosecant	Cotangent
Definition	$\sec x = \frac{1}{\cos x}$	$\operatorname{cosec} x = \frac{1}{\sin x}$	$\cot x = \frac{\cos x}{\sin x}$
Asymptotes	$x = \frac{\pi}{2} + k\pi$ $(90^\circ + 180^\circ k)$	$x = k\pi$ $(180^\circ k)$	$x = k\pi$ $(180^\circ k)$
Range	$y \leq -1$ or $y \geq 1$	$y \leq -1$ or $y \geq 1$	all real y
Period	2π (360°)	2π (360°)	π (180°)
Symmetry	even	odd	odd

Absolute value, sums and differences

Graph	Rule	Check
$y = \lvert f(x) \rvert$	reflect the parts below the x axis upward	never below the axis; corners at crossing zeros
$y = f(\lvert x \rvert)$	keep $x \geq 0$, mirror it into $x < 0$	always even
$y = f(x) \pm g(x)$	add or subtract heights at each x	domain is the intersection of the domains
number of solutions of $F(x) = k$	count meetings of $y = k$ with $y = F(x)$	boundary k values counted separately

Inverse functions

- **One-to-one**: every element of the range corresponds to exactly one element of the domain. Test: no horizontal line meets the graph twice.
- Inverse functions exist for one-to-one functions. If f is not one-to-one, **restrict its domain** until it is (for a parabola, one side of the vertex).

- Reflection in $y = x$ sends (a, b) to (b, a). The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in $y = x$.
- Domain of f^{-1} = range of f; range of f^{-1} = domain of f.
- Formal definition: $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$, each on the appropriate domain.
- $f^{-1}(x)$ is not $\dfrac{1}{f(x)}$.

Recipe: find an inverse

1. Check one-to-one (or note the given restriction).
2. Write $x = f(y)$ by interchanging x and y.
3. Make y the subject. If a ± appears, choose the sign that gives the range of f^{-1}, which is the domain of f.
4. State the domain of f^{-1}: the range of f.

Recipe: where f meets f^{-1}

1. If f is increasing, every meeting point is on $y = x$: solve $f(x) = x$.
2. Reject roots outside the domain of f, with the reason.
3. If f is decreasing, also look for pairs with $f(a) = b$, $f(b) = a$, $a \neq b$.

Parametric form

Curve	Parametric form	Eliminate by
line	$x = x_1 + t, y = y_1 + mt$	substitution: $t = x - x_1$
function $y = f(x)$	$x = t, y = f(t)$	$t = x$
parabola	$x = 2at, y = at^2$ gives $x^2 = 4ay$	$t = \frac{x}{2a}$
circle	$x = a + r\cos t, y = b + r\sin t$	$\cos^2 t + \sin^2 t = 1$

Restrictions: the possible values of x and y in the parametric equations restrict the Cartesian graph. $x = t^2$ forces $x \geq 0$; a parameter interval such as $0 \leq t \leq \pi$gives part of a circle. Check the endpoints by substituting the end values of t.

Inequalities

Type	Method	Watch for
Cubic, linear factors	zeros on a number line; sign on the right from the leading coefficient; alternate (not through a squared factor)	a negative leading coefficient, as in $(2 - x)$
Variable in the denominator	multiply by the denominator squared, factorise, solve; or one fraction and a sign diagram	exclude the zero of the denominator
$\lvert ax + b\rvert < k\ (k > 0)$	$-k < ax + b < k$	dividing by a negative reverses both signs
$\lvert ax + b\rvert > k\ (k > 0)$	$ax + b < -k$ or $ax + b > k$	never $ax + b > \pm k$
$\lvert ax + b\rvert$ with $k < 0$	less than: no solution; greater than: all real x	$\lvert x - c\rvert$ is the distance from x to c

The traps, and can I do it

1. Writing $f^{-1}(x)$ for the reciprocal $\frac{1}{f(x)}$.
2. Forgetting that a horizontal asymptote $y = c$ of f becomes $y = \frac{1}{c}$ on the reciprocal.
3. Putting asymptotes of $\cot x$ at the asymptotes of $\tan x$ instead of at its zeros.
4. For $f(|x|)$, reflecting the left half instead of the right half.
5. Adding ranges to get the range of a sum.
6. Missing a boundary value of k when counting solutions.
7. Keeping both roots of $\pm\sqrt{\ }$ when finding an inverse, or both roots when solving $f(x) = x$.
8. Using the $y = x$ shortcut for a decreasing function.
9. Dropping the restriction after eliminating a parameter.
10. Multiplying an inequality by an expression containing x, or writing $ax + b > \pm k$.

Tick each one you can do without notes. Anything unticked sends you back to the Topic Notes section whose number is in brackets.

- ☐ Sketch $y = \frac{1}{f(x)}$ for a quadratic and for an exponential, with every asymptote (1)
- ☐ Sketch $\sec x$, $\operatorname{cosec} x$, $\cot x$ in radians and degrees and state their properties (2)
- ☐ Write $|f(x)|$ and $f(|x|)$ piecewise and sketch both (3)
- ☐ Find the domain and range of a sum or a difference such as $x + \frac{1}{x}$ (4)
- ☐ Count the solutions of $F(x) = k$ for every k (5)
- ☐ Find an inverse with a restricted domain, choosing the sign with a reason (6)
- ☐ Find where a function meets its inverse, rejecting roots properly (6)
- ☐ Convert a line, parabola and circle between parametric and Cartesian form with restrictions (7)
- ☐ Solve cubic, rational and absolute value inequalities and test the answer (8)

Written to the NSW Mathematics Extension 1 11-12 Syllabus (2024), the syllabus Year 11 has studied since 2026, first examined in the 2027 HSC. Students in Year 11 in 2027 sit the HSC in 2028.

About this resource

MA99 EDUCATION

Scope

NSW Mathematics Extension 1, Year 11. Written to the NSW Mathematics Extension 1 11-12 Syllabus (2024), focus area 1, Further work with functions (ME1-11-01), for students sitting the HSC in 2028 and later years. Every value, identity and solution was re-derived independently by computer algebra before printing. Contexts in the questions are invented unless a source is named.

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MA99 EDUCATION

Focus area 1 Further work with functions: Practice Questions

01

NSW MATHEMATICS EXTENSION 1 11-12 SYLLABUS 2024 FOR HSC 2028 AND LATER

How to use this set

Band	Questions	Marks
Foundation	1 to 7	10
Standard	8 to 11	10
Advanced	12 to 15	13
Exam-style	16 and 17	12

Total 45 marks. Questions 1 to 4 are multiple choice. Every content group is in the Foundation band and again in at least one later band. Attempt a question in full before reading its solution, and for every inequality test one value of your answer in the original.

Questions

FOUNDATION One idea, applied once.

Question 1

1 MARK

Which of the following are the vertical asymptotes of $y = \dfrac{1}{x^2 - 9}$?

A $x = 9$

B $x = -3$ and $x = 3$

C $y = 0$

D $x = 0$ and $y = 0$

Question 2

1 MARK

Which statement about $y = \cot x$ is true?

A It has period 2π.

B It is an even function.

C Its range is $y \leq -1$or $y \geq 1$.

D It has vertical asymptotes $x = k\pi$ for every integer k.

Question 3

1 MARK

What is the solution of $|2x - 1| < 3$?

A $-1 < x < 2$

B $x < 2$

C $-2 < x < 1$

D $x < -1$ or $x > 2$

Question 4

1 MARK

A curve has parametric equations $x = 2t, y = t^2 - 1$. What is its Cartesian equation?

A $y = 4x^2 - 1$

B $y = \frac{x^2}{2} - 1$

C $y = \frac{x^2}{4} - 1$

D $y = \frac{x^2-1}{4}$

5 Find the inverse of $f(x) = 3x - 6$ and verify that $f\left(f^{-1}(x)\right) = x$. (2 marks)

6 Solve $(x-1)(x+2)(x-5) > 0$. (2 marks)

7 For $f(x) = x^2 - 4x$, write $f(|x|)$ in terms of x and find all of its x intercepts. (2 marks)

STANDARD Two steps, or one step plus a decision.

8 Solve $\dfrac{2}{x+1} \leq 1$. (3 marks)

9 State the equations of all asymptotes of $y = \dfrac{1}{x^2 - 4x}$ and the coordinates of its turning point. (2 marks)

10 Solve $|3 - 2x| \geq 7$ (2 marks)

11 A curve has parametric equations $x = 1 + 3\cos\theta$, $y = -2 + 3\sin\theta, 0 \leq \theta \leq \pi$ Find its Cartesian equation and describe the curve, including its endpoints. (3 marks)

ADVANCED Three or more steps, or a method you must choose first.

12 For $y = \operatorname{cosec} x, -360° \leq x \leq 360°$, state the domain, the range and the equations of the vertical asymptotes, and explain why the graph has no x intercepts. (3 marks)

13 Let $f(x) = \dfrac{x+2}{x-1}$. **(a)** Find $f^{-1}(x)$. **(b)** State the domain and range of f^{-1}, and explain what part (a) says about the graph of $y = f(x)$. (4 marks)

14 Find the domain and range of $y = \sqrt{x} - \sqrt{x-1}$. (3 marks)

15 Solve $\dfrac{x}{x-2} > x$. (3 marks)

EXAM-STYLE Multi-part, written the way the HSC writes them.

Question 16

6 MARKS

Let $f(x) = (x-2)^2 + 1$.

(a) State the largest domain of the form $x \geq a$ on which f has an inverse. 1 MARK

(b) For this domain, find $f^{-1}(x)$ and state its domain. 2 MARKS

(c) Find the exact coordinates of any point where $y = f(x)$ meets $y = f^{-1}(x)$. 2 MARKS

(d) Explain why every such point lies on the line $y = x$. 1 MARK

Question 17

6 MARKS

Let $f(x) = x^2 - 4x + 3$.

(a) Sketch $y = |f(x)|$, showing the intercepts and the coordinates of any turning point. 2 MARKS

(b) Hence find the values of k for which $|f(x)| = k$ has exactly four real solutions. 2 MARKS

(c) Find the values of k for which $\dfrac{1}{f(x)} = k$ has no real solutions. 2 MARKS

WORKING SPACE

Worked solutions · Foundation

1

(B) $x^2 - 9 = (x - 3)(x + 3)$ is zero at $x = \pm 3$, so the reciprocal has vertical asymptotes $x = -3$ and $x = 3$.

(A) treats the zero of $x^2 - 9$ as $x = 9$. (C) is the horizontal asymptote, not a vertical one. (D) is the asymptotes of $y = \frac{1}{x}$, copied across without looking at the denominator.

2

(D) $\cot x = \frac{\cos x}{\sin x}$ is undefined exactly where $\sin x = 0$, at $x = k\pi$.

(A) is the period of $\sec x$ and $\operatorname{cosec} x$; $\cot x$ has period π. (B) is false: $\cot(-x) = -\cot x$, so it is odd. (C) is the range of $\sec x$ and $\operatorname{cosec} x$; $\cot x$ takes every real value.

3

(A) $-3 < 2x - 1 < 3$, so $-2 < 2x < 4$ and $-1 < x < 2$.

(B) solves only $2x - 1 < 3$ and drops the lower bound. (C) solves $|2x + 1| < 3$, a sign slip inside the absolute value. (D) is the solution of $|2x - 1| > 3$, the outside instead of the between.

4

(C) $t = \frac{x}{2}$, so $y = \left(\frac{x}{2}\right)^2 - 1 = \frac{x^2}{4} - 1$.

(A) substitutes $t = 2x$, inverting the relationship. (B) squares $\frac{x}{2}$ as $\frac{x^2}{2}$. (D) divides the whole of $x^2 - 1$ by 4, which is $\frac{x^2}{4} - \frac{1}{4}$.

5

Interchange: $x = 3y - 6$, so $y = \dfrac{x+6}{3}$. $f^{-1}(x) = \dfrac{x+6}{3}$.

Check: $f\left(f^{-1}(x)\right) = 3 \cdot \dfrac{x+6}{3} - 6 = x + 6 - 6 = x$.

WHERE THE MARKS ARE	
1 mark	Correct inverse
1 mark	Composition simplified to x

6

Zeros $-2, 1, 5$, leading term x^3 positive: positive for $x > 5$, negative on $(1, 5)$, positive on $(-2, 1)$, negative for $x < -2$. So $-2 < x < 1$ or $x > 5$.

WHERE THE MARKS ARE	
1 mark	Zeros and the sign pattern
1 mark	Both intervals, strict

7

$f(|x|) = x^2 - 4|x|$, since $|x|^2 = x^2$. Intercepts: $|x|\big(|x| - 4\big) = 0$, so $|x| = 0$ or $|x| = 4$: $x = -4, 0, 4$.

WHERE THE MARKS ARE	
1 mark	$x^2 - 4\|x\|$
1 mark	All three intercepts

f itself has intercepts 0 and 4. The even graph $f(|x|)$ gains the mirror image -4.

Worked solutions · Standard

8

$x \neq -1$. Multiply by $(x+1)^2 > 0$:

$$2(x+1) \leq (x+1)^2$$

$$(x+1)^2 - 2(x+1) \geq 0 \implies (x+1)(x-1) \geq 0 \quad \text{factorise, do not expand}$$

$$x \leq -1 \ \text{ or } \ x \geq 1 \quad \text{outside the zeros of an upward parabola}$$

Exclude $x = -1$: $x < -1$ or $x \geq 1$.

WHERE THE MARKS ARE	
1 mark	Multiplies by $(x+1)^2$ or uses a valid one-fraction method
1 mark	$(x+1)(x-1) \geq 0$
1 mark	Answer with -1 excluded and 1 included

Test: $x = -2$ gives $-2 \leq 1$, true; $x = 0$ gives $2 \leq 1$, false; $x = 1$ gives $1 \leq 1$, true. The answer survives all three.

9

$x^2 - 4x = x(x-4)$: vertical asymptotes $x = 0$ and $x = 4$, and $\frac{1}{x^2-4x} \to 0$ as $x \to \pm\infty$ so $y = 0$ is a horizontal asymptote.

$x^2 - 4x = (x-2)^2 - 4$ has minimum -4 at $x = 2$, so the reciprocal has a local maximum $\left(2, -\frac{1}{4}\right)$.

WHERE THE MARKS ARE	
1 mark	All three asymptotes
1 mark	Turning point

10

$3 - 2x \geq 7$ gives $-2x \geq 4$ $x \leq -2$ $3 - 2x \leq -7$ gives $-2x \leq -10$ $x \geq 5$ $x \leq -2$ or $x \geq 5$.

WHERE THE MARKS ARE	
1 mark	Both cases set up, joined by or
1 mark	Both solved with the signs reversed

11

$\cos\theta = \frac{x-1}{3}$, $\sin\theta = \frac{y+2}{3}$, so $(x-1)^2 + (y+2)^2 = 9$: the circle with centre $(1, -2)$ and radius 3.

For $0 \leq \theta \leq \pi$ $\sin\theta \geq 0$ so $y \geq -2$ the upper semicircle, from $(4, -2)$ at $\theta = 0$ through $(1, 1)$ to $(-2, -2)$ at $\theta = \pi$.

WHERE THE MARKS ARE	
1 mark	Cartesian equation
1 mark	Centre and radius
1 mark	Upper half with both endpoints

Worked solutions · Advanced

12

$\sin x = 0$ at $x = -360°, -180°, 0°, 180°, 360°$: these are the vertical asymptotes.

Domain: $-360° < x < 360°, x \neq -180°, 0°, 180°$. Range: $y \leq -1$ or $y \geq 1$.

No x intercepts: $\operatorname{cosec} x = \frac{1}{\sin x}$, and a fraction with numerator 1 is never 0. In fact $|\sin x| \leq 1$ gives $|\operatorname{cosec} x| \geq 1$.

WHERE THE MARKS ARE	
1 mark	All five asymptotes
1 mark	Domain and range
1 mark	The reason

13

(a) Interchange: $x = \frac{y+2}{y-1}$, so $xy - x = y + 2$, $y(x-1) = x + 2$, and $f^{-1}(x) = \dfrac{x+2}{x-1}$, the same rule as f.

(b) $f(x) = 1 + \frac{3}{x-1}$ has domain $x \neq 1$ and range $y \neq 1$. So f^{-1} has domain $x \neq 1$, range $y \neq 1$. Since $f^{-1} = f$, the graph of $y = f(x)$ is its own reflection in $y = x$: it is symmetric about the line $y = x$.

Check: $f(f(x)) = \dfrac{\frac{x+2}{x-1} + 2}{\frac{x+2}{x-1} - 1} = \dfrac{3x}{3} = x$.

WHERE THE MARKS ARE	
1 mark	Interchanges and collects y
1 mark	$f^{-1}(x) = \frac{x+2}{x-1}$
1 mark	Domain and range
1 mark	Symmetry in $y = x$ explained

14

Domain: $x \geq 0$ and $x \geq 1$, so $x \geq 1$.

Rationalise: $\sqrt{x} - \sqrt{x-1} = \dfrac{x - (x-1)}{\sqrt{x} + \sqrt{x-1}} = \dfrac{1}{\sqrt{x} + \sqrt{x-1}}$.

The denominator is 1 at $x = 1$ and increases without bound, so y starts at 1 and decreases towards 0 without reaching it. Range $0 < y \leq 1$.

WHERE THE MARKS ARE	
1 mark	Domain
1 mark	Rationalised form or another valid argument
1 mark	Range, with 0 excluded

The difference of two increasing functions can decrease. This is a question about the difference, not about the two terms separately.

15

$x \neq 2$. Multiply by $(x-2)^2 > 0$:

$$x(x-2) > x(x-2)^2$$

$$x(x-2)\big[1 - (x-2)\big] > 0 \implies x(x-2)(3-x) > 0 \quad \text{common factor}$$

$$\text{leading term } -x^3 : \text{ negative for } x > 3 \quad \text{zeros } 0, 2, 3$$

Positive on $(2, 3)$ and for $x < 0$: $x < 0$ or $2 < x < 3$.

WHERE THE MARKS ARE	
1 mark	Multiplies by $(x-2)^2$
1 mark	$x(x-2)(3-x)>0$
1 mark	Both intervals

Test: $x=-1$ gives $\frac{1}{3}>-1$, true; $x=1$ gives $-1>1$, false; $x=2.5$ gives $5>2.5$, true; $x=4$ gives $2>4$, false.

Worked solutions · Exam-style

16

(a) The vertex is $(2,1)$, so f is increasing on $x \geq 2$ and one-to-one there.

(b) The range is $y \geq 1$ Interchange: $x=(y-2)^2+1$, $y-2=\pm\sqrt{x-1}$. The range of f^{-1} is $y \geq 2$, so $f^{-1}(x)=2+\sqrt{x-1}$, domain $x \geq 1$.

(c) Solve $f(x)=x$: $x^2-4x+5=x$, $x^2-5x+5=0$, $x=\frac{5\pm\sqrt{5}}{2}$. $\frac{5-\sqrt{5}}{2} \approx 1.38<2$ is outside the domain of f. The point is $\left(\frac{5+\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right)$.

(d) f is increasing. If (a,b) lies on both graphs then $f(a)=b$ and $f(b)=a$. If $a<b$, increasing gives $f(a)<f(b)$, that is $b<a$, a contradiction; $a>b$ fails the same way. So $a=b$, and the point is on $y=x$.

WHERE THE MARKS ARE	
1 mark	(a) $x \geq 2$
1 mark	(b) $2+\sqrt{x-1}$ with the sign justified
1 mark	(b) domain $x \geq 1$
1 mark	(c) solves $f(x)=x$
1 mark	(c) rejects the smaller root with a reason
1 mark	(d) the increasing argument

17

(a) $f(x)=(x-1)(x-3)=(x-2)^2-1$. Reflect the part on $1<x<3$ upward: corners on the axis at $(1,0)$ and $(3,0)$, y intercept $(0,3)$, and the vertex $(2,-1)$ becomes a local maximum $(2,1)$. Outside $[1,3]$ the arms rise without bound.

(b) A horizontal line $y=k$ meets both arms and the arch twice when it is above the axis and below the top of the arch: $0<k<1$. At $k=1$ there are three solutions, and at $k=0$ two.

(c) f takes the values $[-1,0)$ on $1<x<3$ and every positive value outside, so $\frac{1}{f}$ takes every value in $(-\infty,-1]$ and in $(0,\infty)$ It never equals k for $-1<k \leq 0$.

WHERE THE MARKS ARE

1 mark	(a) shape with the corners
1 mark	(a) local maximum $(2, 1)$ and intercept
1 mark	(b) uses the sketch, hence
1 mark	(b) $0 < k < 1$, both ends strict
1 mark	(c) range of $\frac{1}{f}$
1 mark	(c) $-1 < k \leq 0$

In (c), $k = 0$ belongs in the answer: a reciprocal is never zero. $k = -1$ does not: $\frac{1}{f} = -1$ at the vertex, $x = 2$.

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