CHAPTER 2 · FUNCTIONS · MAV-11-03

2. Graph Transformations

Almost every graph in this course is a parent graph that has been moved, flipped or stretched. Learn what each change to an equation does to the picture, and one sketch of $y = f(x)$ becomes the sketch of a whole family.

Year 11 Mathematics Advanced · written to the NSW Mathematics Advanced 11-12 Syllabus (2024), focus area Graph transformations.

What changed in 2024

This content is new to Year 11. Under the 2017 syllabus, Year 11 met the reflections $y = -f(x)$, $y = f(-x)$ and $y = -f(-x)$ and a few transformed exponential and logarithmic graphs. The general work on translations, dilations and combined transformations was Year 12 content. NESA moved it into Year 11 with the 2024 syllabus.

The 2024 content also names three ideas explicitly: describing a dilation as a stretch from an axis, the **enlargement** and **reduction** (both variables dilated by the same factor), and the warning that the order of transformations matters. Circles of the form $x^2 + y^2 + ax + by + c = 0$, found by completing the squares, are part of this focus area.

Trigonometric graphs are excluded here. Transforming them is Year 12 work, in Further graph transformations and modelling.

This chapter builds on Chapter 1 and on Stage 5: the parent graphs it transforms are ones you have already met. The exponential $y = 2^x$ is developed in full in Chapter 6, and the logarithmic graphs of Chapter 6 are transformed by exactly the same rules once you have them.

2.1 Parent graphs and the replacement rule

A **transformation** changes the position, size or orientation of a graph. The graphs being transformed are the parent graphs: a small set of shapes whose key points you should know without thinking. Every graph in this chapter is one of these, moved.

Parent	Equation	Domain	Range	Key features
Parabola	$y = x^2$	all real x	$y \geq 0$	vertex $(0, 0)$; $(\pm 1, 1)$, $(\pm 2, 4)$
Cubic	$y = x^3$	all real x	all real y	$(0, 0)$, $(1, 1)$, $(-1, -1)$
Hyperbola	$y = \dfrac{1}{x}$	$x \neq 0$	$y \neq 0$	asymptotes $x = 0, y = 0$; $(1, 1)$, $(-1, -1)$
Square root	$y = \sqrt{x}$	$x \geq 0$	$y \geq 0$	endpoint $(0, 0)$; $(1, 1)$, $(4, 2)$
Absolute value	$y = \lvert x \rvert$	all real x	$y \geq 0$	vertex $(0, 0)$; $(\pm 1, 1)$
Exponential	$y = 2^x$	all real x	$y > 0$	asymptote $y = 0$; $(0, 1)$, $(1, 2)$
Circle	$x^2 + y^2 = r^2$	$-r \leq x \leq r$	$-r \leq y \leq r$	centre $(0, 0)$, radius r

The replacement rule

Every transformation in this chapter is made by **replacing** x, or y, or both, in the equation of a graph by a new expression. Nothing else in the equation changes.

The replacement always does the opposite of what it seems to say. Replacing x by $x - 3$ moves the graph 3 units to the **right**. Replacing x by $2x$ squeezes it to **half** its width.

The reason takes one line. A point (X, Y) is on $y = f(x - 3)$ exactly when $Y = f(X - 3)$, that is, when $(X - 3, Y)$ is on $y = f(x)$. So each point of the new graph is a point (p, q) of the old graph with 3 added to its x coordinate: $(p, q) \to (p + 3, q)$. The equation undoes the move, so to move right you subtract, and to stretch you divide.

The rule is stated for the equation of a graph, not only for $y = f(x)$, so it works just as well for relations such as circles, where y is not the subject. That is why NESA states every rule in this focus area as a replacement of x or of y, and why this chapter does too.

Investigate with a graphing application

NESA asks you to establish each rule with a graphing application before you rely on it. One session with sliders does the lot.

1. Define $f(x) = x^2$. **2.** Enter $y = f(x - a)$ with a slider for a from -5 to 5 and watch which way the graph moves as a increases. **3.** Do the same for $y - b = f(x)$, $y = f(-x)$, $-y = f(x)$, $y = f\left(\frac{x}{k}\right)$ and $\frac{y}{l} = f(x)$. **4.** Change f to $\sqrt{x}$, $\frac{1}{x}$ or 2^x and check that every conclusion still holds.

2.2 Translations

A **translation** slides every point of a graph the same distance in the same direction. The shape, size and orientation are unchanged; only the position moves.

Translations

Replacing x by $x - a$ is a **horizontal translation** by a: to the right when $a > 0$, to the left when $a < 0$. For a function, $y = f(x)$ becomes $y = f(x - a)$.

Replacing y by $y - b$ is a **vertical translation** by b: up when $b > 0$, down when $b < 0$. For a function, $y - b = f(x)$ rearranges to $y = f(x) + b$.

Every point moves by the same amount: $(p, q) \to (p + a,\ q + b)$. Asymptotes, vertices and endpoints move with the graph.

The **sign** of a and of b decides the direction, and it is read from the replacement, not from how the equation looks. $y = f(x + 2)$ is x replaced by $x - (-2)$, so $a = -2$ and the graph moves 2 units **left**. $y = f(x) - 3$ is $y + 3 = f(x)$, which is y replaced by $y - (-3)$, so $b = -3$ and the graph moves 3 units **down**.

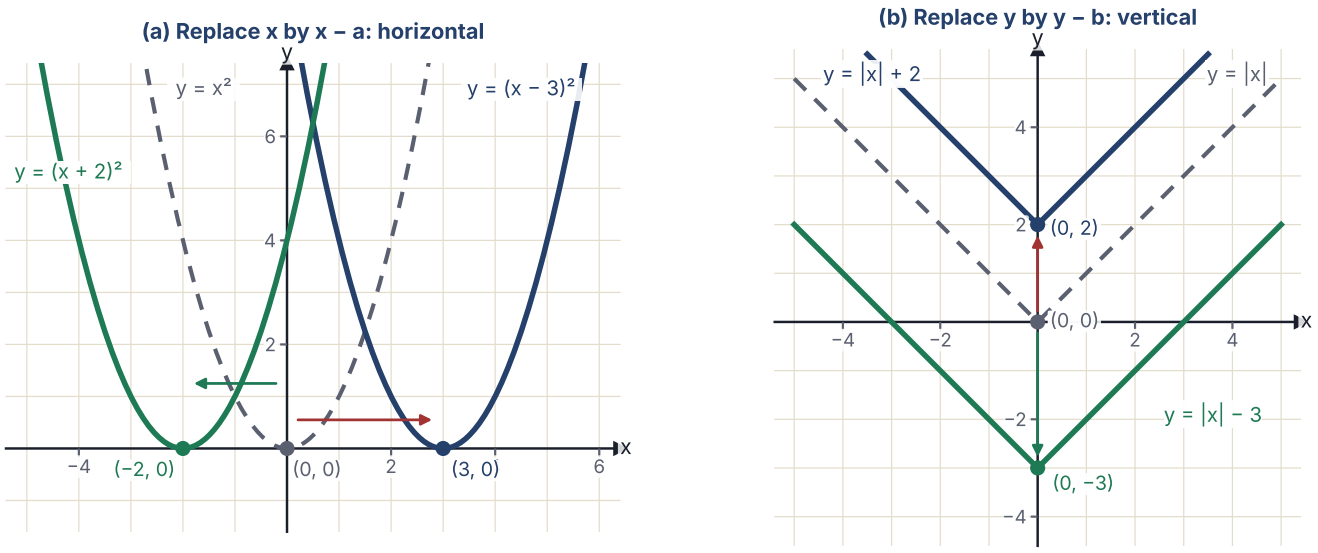


Figure 2.1 · (a) Replacing x by $x - 3$ moves $y = x^2$ three units right; replacing x by $x + 2$ moves it two units left. (b) Replacing y by $y - 2$ moves $y = |x|$ up 2; replacing y by $y + 3$ moves it down 3. Parents are dashed throughout this chapter.

Worked example 1

3 MARKS

Describe the transformations that map $y = x^2$ onto $y = (x + 4)^2 - 1$. State the vertex and the range of the image.

01 x has been replaced by $x + 4 = x - (-4)$, so $a = -4$: a horizontal translation 4 units **left**. the sign inside is opposite to the move

02 $y = (x + 4)^2 - 1$ is $y + 1 = (x + 4)^2$, so y has been replaced by $y - (-1)$ and $b = -1$: a vertical translation 1 unit **down**.

03 The vertex $(0, 0)$ moves to $(0 - 4,\ 0 - 1) = (-4, -1)$.

04 The range of $y = x^2$ is $y \geq 0$ and every y value drops by 1, so the range is $y \geq -1$, that is $[-1, \infty)$

Name the direction.
Left, right, up or down. A translation without a direction is half an answer.

Move the key point.
The vertex carries the whole parabola. Move it and the range follows.

WHERE THE MARKS ARE

1 for the horizontal translation with its direction, **1** for the vertical translation, **1** for the vertex and the range. 'Translated 4 units' with no direction is not a description and earns nothing for that part.

Worked example 2

4 MARKS

The graph of $y = \dfrac{1}{x}$ is translated 2 units right and then 1 unit up. Find the equation of the image, its asymptotes, its intercepts with the axes, and its domain and range.

01 Right 2: replace x by $x - 2$, giving $y = \dfrac{1}{x - 2}$.

02 Up 1: replace y by $y - 1$, giving $y - 1 = \dfrac{1}{x - 2}$, so $y = \dfrac{1}{x - 2} + 1$.

03 The asymptotes $x = 0$ and $y = 0$ move with the graph, to $x = 2$ and $y = 1$.
asymptotes translate like any other line

04 y intercept: $x = 0$ gives $y = -\frac{1}{2} + 1 = \frac{1}{2}$. x intercept: $\frac{1}{x-2} = -1$, so $x - 2 = -1$ and $x = 1$. intercepts are recalculated, not moved

05 Domain $x \neq 2$, range $y \neq 1$.

Asymptotes move.
They are lines in the picture. Translate them with everything else.

Intercepts do not.
The axes stay still while the graph moves. Substitute x = 0 and y = 0 again.

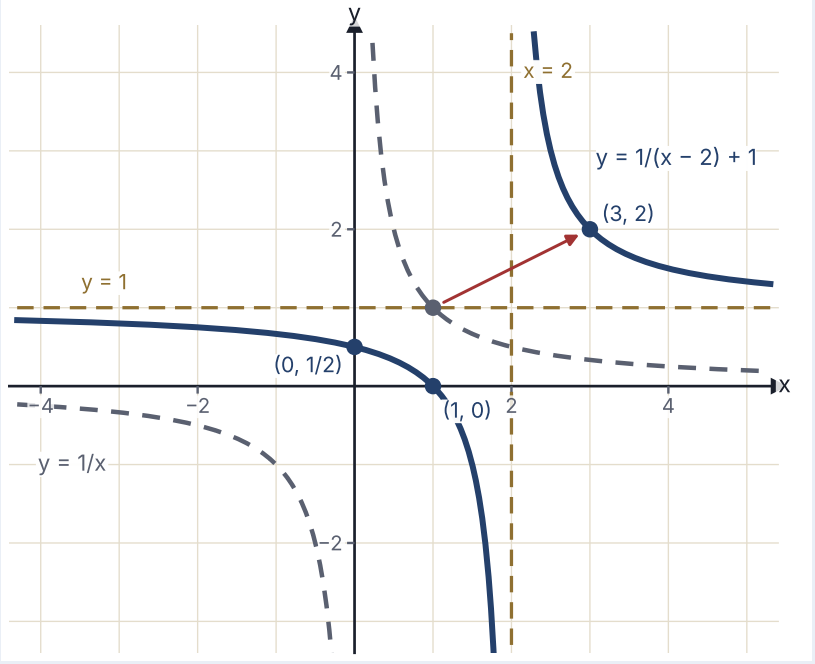


Figure 2.2 · $y = \frac{1}{x-2} + 1$ (solid) is $y = \frac{1}{x}$ (dashed) moved 2 right and 1 up. The point $(1, 1)$ goes to $(3, 2)$ and the asymptotes go to $x = 2$ and $y = 1$. The intercepts $(1, 0)$ and $(0, \frac{1}{2})$ come from the new equation: $y = \frac{1}{x}$ has no intercepts to move.

WHERE THE MARKS ARE

1 for the equation, **1** for both asymptotes, **1** for both intercepts, **1** for the domain and range. The intercepts must be calculated from the new equation.

The trap · the sign of the translation

$y = (x + 4)^2$ moves **left**, and $y = f(x) - 3$ moves **down**. The vertical case only looks 'the way it reads' because the -3 has been carried to the other side: as a replacement it is $y + 3 = f(x)$, and the sign flips exactly as it does inside the bracket. Learn the replacement rule and you have one rule, not two.

2.3 Reflections

A **reflection** creates the mirror image of a graph in a line. In this course the mirror is one of the coordinate axes.

Reflections

Replacing x by $-x$ reflects the graph in the y**-axis**: $y = f(x)$ becomes $y = f(-x)$, and $(p, q) \to (-p, q)$.

Replacing y by $-y$ reflects the graph in the x**-axis**: $-y = f(x)$, that is $y = -f(x)$, and $(p, q) \to (p, -q)$.

Doing both gives $y = -f(-x)$ and $(p, q) \to (-p, -q)$: a **rotation of** $180°$ **about the origin**. The two reflections can be done in either order.

The mirror is the axis the replaced variable does **not** measure. Change x and every point moves horizontally, across the vertical axis. Change y and every point moves vertically, across the horizontal axis. Points on the mirror stay where they are, so $y = f(-x)$ keeps the y intercept and $y = -f(x)$ keeps every x intercept.

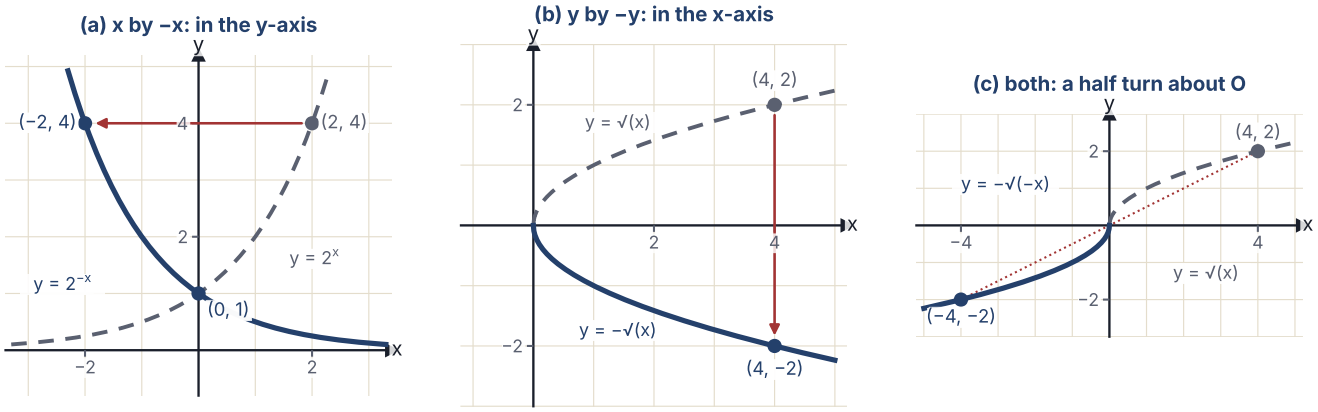


Figure 2.3 · (a) Replacing x by $-x$ reflects $y = 2^x$ in the y-axis: $(2, 4)$ goes to $(-2, 4)$ and $(0, 1)$, on the mirror, stays. (b) Replacing y by $-y$ reflects $y = \sqrt{x}$ in the x-axis: $(4, 2)$ goes to $(4, -2)$. (c) Both replacements rotate $y = \sqrt{x}$ through 180° about O: $(4, 2)$ goes to $(-4, -2)$.

Chapter 1 defined even and odd functions by exactly these transformations. A function is **even** when replacing x by $-x$ leaves its graph unchanged, $f(-x) = f(x)$. It is **odd** when the half turn $y = -f(-x)$ leaves it unchanged, $-f(-x) = f(x)$, which rearranges to $f(-x) = -f(x)$.

Worked example 3

3 MARKS

Given $f(x) = x^2 - 2x - 3$, find the equations of $y = f(-x)$ and $y = -f(x)$, and the x intercepts and vertex of each graph.

01 $f(x) = (x - 3)(x + 1) = (x - 1)^2 - 4$: x intercepts 3 and -1, vertex $(1, -4)$. know the features you are moving

02 $f(-x) = (-x)^2 - 2(-x) - 3 = x^2 + 2x - 3 = (x + 3)(x - 1)$: intercepts -3 and 1, vertex $(-1, -4)$. brackets round $-x$ every time

03 $-f(x) = -x^2 + 2x + 3$: the same intercepts 3 and -1, vertex $(1, 4)$.

04 Check against the rules. The reflection in the y-axis negated the intercepts and the vertex's x coordinate. The reflection in the x-axis kept the intercepts and negated the vertex's y coordinate.

Bracket the −x.
$(-x)^2$ is x^2. Without the bracket the sign goes wrong.

Check the mirror.
Intercepts that sit on the mirror line never move.

WHERE THE MARKS ARE

1 for each equation with correct signs, **1** for the intercepts and vertices. Writing $f(-x) = -x^2 + 2x - 3$, by leaving out the brackets round $-x$, is the usual loss.

Worked example 4

2 MARKS

Show that the graph of $y = x^3 - 4x$ is unchanged by a rotation of 180° about the origin.

01 The rotation is $y = -f(-x)$ with $f(x) = x^3 - 4x$.

02 $-f(-x) = -\left[(-x)^3 - 4(-x)\right]$ two minus signs: one inside, one outside

03 $= -\left[-x^3 + 4x\right] = x^3 - 4x = f(x)$.

04 The image has the same equation, so the graph is unchanged, and f is odd.

Two signs.
One minus inside for the y-axis, one outside for the x-axis.

Conclude in words.
Unchanged by a half turn means odd. Say so.

WHERE THE MARKS ARE

1 for forming $-f(-x)$ with correct brackets, **1** for simplifying it to $f(x)$ and stating the conclusion.

The trap · which axis is the mirror

$y = f(-x)$ is the reflection in the **y-axis**. Students reach for the x-axis because x is the letter that changes. Settle it with one point: $(2, 4)$ on $y = 2^x$ becomes $(-2, 4)$ on $y = 2^{-x}$, which is across the y-axis.

Exercise 2A · Translations and reflections

FOUNDATION One idea, applied once. The point of them is to check the idea is there.

1 Describe the transformation that maps $y = x^2$ onto $y = (x - 5)^2$. (1 mark)

2 Describe the transformation that maps $y = |x|$ onto $y = |x| - 4$. (1 mark)

3 The graph of $y = \frac{1}{x}$ is translated 3 units left. Write its equation and state its vertical asymptote. (2 marks)

4 Write the equations of the reflections of $y = 2^x$ in the y-axis and in the x-axis. (2 marks)

STANDARD Two steps, or one step plus a decision. Most of the examination lives here.

5 $y = x^2$ is translated 2 units left and 3 units down. Write the equation of the image and state its vertex and y intercept. (3 marks)

6 $f(x) = x^2 - 6x + 5$. Find $f(-x)$ and $-f(x)$, and the x intercepts of $y = f(-x)$ and of $y = -f(x)$. (3 marks)

7 The point $(3, -2)$ lies on $y = f(x)$. Find its image on (a) $y = f(x + 4)$ (b) $y = f(x) + 4$ (c) $y = -f(-x)$. (3 marks)

8 Show that replacing x by $-x$ in $y = x^4 - 3x^2$ leaves the equation unchanged, and explain what this says about the graph. (2 marks)

ADVANCED Three or more steps, or a method you must choose before you can start.

9 The graph of $y = \sqrt{x}$ is translated so that its endpoint moves from $(0, 0)$ to $(-3, 2)$. Write the equation of the image and state its domain and range. (3 marks)

10 The graph of $y = \frac{1}{x}$ is translated so that its asymptotes are $x = -1$ and $y = 3$. Write its equation and find its intercepts with the axes. (4 marks)

11 Show that the graph of $y = x^3 - 3x$ is unchanged by the transformation $y = -f(-x)$, and name the symmetry this shows. (3 marks)

12 The parabola $y = x^2 - 4x + 1$ is translated 3 units right and 2 units up. Find the equation of the image in expanded form, and its vertex. (4 marks)

EXAM-STYLE Multi-part, written the way the HSC writes them. Attempt these last.

13 (a) Show that $y = x^2 + 4x + 7$ can be written as $y = (x + 2)^2 + 3$. (b) Hence describe the translations that map $y = x^2$ onto $y = x^2 + 4x + 7$. (c) State the range of $y = x^2 + 4x + 7$. (4 marks)

14 (a) $y = 2^x$ is reflected in the y-axis and then translated 1 unit down. Write the equation of the image. (b) State its asymptote, its intercepts and its range. (c) Explain why translating first and reflecting second gives the same graph here. (5 marks)

2.4 Dilations: stretching from an axis

A **dilation** stretches a graph away from an axis, or compresses it toward one. Use NESA's description in an answer: a **horizontal dilation** is "a stretch from the y-axis", and a **vertical dilation** is "a stretch from the x-axis" (NESA). Every point moves parallel to one axis, and its distance from the other axis is multiplied by the **dilation factor**.

Dilations

Replacing x by $\dfrac{x}{k}$ is a **horizontal dilation by a factor of k**, a stretch from the y-axis: $y = f\left(\dfrac{x}{k}\right)$, and $(p, q) \to (kp,\ q)$.

Replacing y by $\dfrac{y}{l}$ is a **vertical dilation by a factor of l**, a stretch from the x-axis: $\dfrac{y}{l} = f(x)$, that is $y = l\,f(x)$, and $(p, q) \to (p,\ lq)$.

A factor greater than 1 stretches the graph away from the axis. A factor between 0 and 1 compresses it toward the axis. Points on the axis it is stretched from do not move.