MA99 EDUCATION

Focus area 5 The binomial theorem: Summary

05

NSW MATHEMATICS EXTENSION 1 11-12 SYLLABUS 2024 FOR HSC 2028 AND LATER

The binomial theorem, on three pages

The words

Term	Meaning
Binomial (expression)	An expression with two terms: $x+y$, $2x-3$, $x^2+\frac{1}{x}$
Binomial expansion	A power of a binomial written as a sum of terms
Binomial coefficient	${}^nC_r = \binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$, the coefficient of $x^{n-r}y^r$ in $(x+y)^n$
Pascal's triangle	Row n (top row is row 0) lists ${}^nC_0, {}^nC_1, \ldots, {}^nC_n$
General term	${}^nC_r\, x^{n-r}y^r$, the $(r+1)$th term, often written T_{r+1}
Constant term	The term independent of x: the term whose power of x is 0
Identity	A statement true for all values of its variable(s)

Pascal's triangle to row 8

Row	Entries	Row sum
0	1	1
1	1, 1	2
2	1, 2, 1	4
3	1, 3, 3, 1	8
4	1, 4, 6, 4, 1	16
5	1, 5, 10, 10, 5, 1	32
6	1, 6, 15, 20, 15, 6, 1	64
7	1, 7, 21, 35, 35, 21, 7, 1	128
8	1, 8, 28, 56, 70, 56, 28, 8, 1	256

Every formula and result

Result	Statement	Condition or note
Binomial theorem	$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}y^r$	n a positive integer; on the reference sheet
General term	${}^nC_r\, x^{n-r}y^r$	the $(r+1)$th term; not on the sheet
With first term 1	$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$	${}^nC_r$ is the coefficient of x^r
Ends	${}^nC_0 = {}^nC_n = 1, {}^nC_1 = {}^nC_{n-1} = n$	

Result	Statement	Condition or note
Symmetry	${}^nC_r = {}^nC_{n-r}$	$0 \le r \le n$
Pascal's rule	${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$	$1 \le r \le n-1$
Row sum	$\sum_{r=0}^{n} {}^nC_r = 2^n$	$x = 1$ in $(1+x)^n$
Alternating sum	$\sum_{r=0}^{n} (-1)^r \, {}^nC_r = 0$	$n \ge 1$; $x = -1$
Even and odd positions	entries in even positions add to 2^{n-1}, and so do those in odd positions	$n \ge 1$; add and subtract the two above
Chair identity	$r \, {}^nC_r = n \, {}^{n-1}C_{r-1}$	$1 \le r \le n$; count committees with a chair
Weighted sum	$\sum_{r=1}^{n} r \, {}^nC_r = n \, 2^{n-1}$	from the chair identity, no calculus
Hockey stick	$\sum_{k=r}^{n} {}^kC_r = {}^{n+1}C_{r+1}$	Pascal's rule, telescoped
Sum of squares	$\sum_{r=0}^{n} ({}^nC_r)^2 = {}^{2n}C_n$	coefficient of x^n in $(1+x)^n(1+x)^n$

Across a row the coefficients rise and then fall, symmetrically. Finding the **greatest** coefficient is not in the 2024 content points.

The methods, as recipes

Task	Recipe
Expand $(a+b)^n$	Row n; falling powers of (a), rising powers of (b), each in brackets with its sign; simplify term by term; check at $x = 1$
Find one term or coefficient	General term with brackets; collect one power of x; set it equal to the target and solve for r; check r is a whole number from 0 to n; evaluate with the sign
Product such as $(a + bx)(1+x)^n$	List each pair of powers adding to the target, and add their products
Unknown k or n	Write one equation per given coefficient; divide or subtract to eliminate
Simplify coefficients	Make r small by symmetry; join next door neighbours in a row by Pascal's rule (the answer sits one row down, larger lower index)
Prove by substitution	Start from $(1+x)^n = \sum {}^nC_r x^r$; choose x to match the weights: try $x = 1$, $x = -1$ or $x = 2$
Prove by comparing coefficients	Write one power as a product of two; find the coefficient of one power of x both ways; equate, giving the reason
Prove by a combinatorial argument	Name the set being counted; count it two ways using the context given; state that both count the same set
Build on an identity	Apply it term by term inside the sum; change the index and both limits; recognise a known sum

Common errors

1. $(x+y)^n \neq x^n + y^n$. The middle terms are the topic.
2. Pascal's triangle starts at row 0, and positions in a row start at 0.
3. The fourth term has $r = 3$. The term containing ${}^nC_r$ is the $(r+1)$th.
4. $(2x)^5 = 32x^5$, not $2x^5$: bracket every term before raising it.
5. A negative second term carries $(-1)^r$. Write it with its sign inside the bracket.
6. No whole number r in $0 \le r \le n$ means there is no such term. Say so, with the reason.

7. A term includes its power of x; a coefficient is the number alone.
8. Pascal's rule joins neighbours in one row only, and needs $1 \le r \le n-1$.
9. Checking values of n is not a proof. A combinatorial proof must say both counts are of the same set.
10. When an index changes in a sum, both limits change.

Checklist

Ten tasks, one per content point, in NESA's order. Tick each one you can do in under two minutes without notes, then check the answers below.

- [ ] Say whether $3x^2 - \sqrt{x}$ is a binomial expression, and why.
- [ ] Write row 7 of Pascal's triangle and check its sum.
- [ ] State the coefficient of x^4y^6 in $(x+y)^{10}$ as a binomial coefficient and evaluate it.
- [ ] Confirm ${}^7C_3 = {}^6C_2 + {}^6C_3$ from the triangle, then prove the general rule two ways.
- [ ] Derive the binomial theorem in four sentences.
- [ ] Expand $(1-2x)^5$.
- [ ] Find the term independent of x in $\left(x^2 + \dfrac{2}{x}\right)^6$.
- [ ] Write ${}^9C_5 + {}^9C_6$ as one coefficient, and ${}^{40}C_{38}$ as a number.
- [ ] Prove $\sum_{r=0}^{n} 2^r\, {}^nC_r = 3^n$.
- [ ] Use $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$ to evaluate $\sum_{r=1}^{8} r\,{}^8C_r$.

Answers: 1 yes, two terms. 2 $1, 7, 21, 35, 35, 21, 7, 1$, sum 128. 3 ${}^{10}C_6 = 210$. 4 $35 = 15 + 20$. 5 see the notes, section 5. 6 $1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$. 7 power x^{12-3r}, $r = 4$, ${}^6C_4\,2^4 = 240$. 8 ${}^{10}C_6 = 210$; 780. 9 put $x = 2$ in $(1+x)^n$. 10 $8 \times 2^7 = 1024$.

On the HSC reference sheet

Given: ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ and the binomial theorem written out. Not given: the general term, the $(1+x)^n$ form, the Pascal identities, and every result proved from them.

> Put in a value, match a coefficient, or count something twice. Every identity is one of the three.

Where this sits

Focus area 5 of the Year 11 Mathematics Extension 1 course, outcome ME1-11-05: “uses the binomial theorem to solve problems and prove identities” (NESA, Mathematics Extension 1 11-12 Syllabus 2024). It is the second focus area in the Combinatorics area of study and builds directly on Permutations and combinations (focus area 4), where ${}^nC_r$ was defined as a number of selections.

It leads to the Year 12 focus area The binomial distribution and sampling distribution of the mean, where the same coefficients give binomial probabilities.

Written to the NSW Mathematics Extension 1 11-12 Syllabus (2024), the syllabus Year 11 has studied since 2026, first examined in the 2027 HSC. Students in Year 11 in 2027 sit the HSC in 2028.

About this resource

MA99 EDUCATION

Scope

NSW Mathematics Extension 1, Year 11. Written to the NSW Mathematics Extension 1 11-12 Syllabus (2024), focus area 5, The binomial theorem (ME1-11-05), for students sitting the HSC in 2028 and later years. Every value, identity and solution was re-derived independently by computer algebra before printing. Contexts in the questions are invented unless a source is named.

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