MA99 EDUCATION

Focus area 2 Further work with vectors: Summary

02

NSW MATHEMATICS EXTENSION 2 11-12 SYLLABUS 2024 FOR HSC 2027 AND LATER

Further work with vectors, on two pages

The formulas

Quantity	Formula	Note
Magnitude	$\lvert\mathbf{a}\rvert = \sqrt{a_1^2 + a_2^2 + a_3^2}$	Leave in exact surd form
Unit vector	$\hat{\mathbf{a}} = \mathbf{a}/\lvert\mathbf{a}\rvert$	
Vector between points	$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$	**This order.** Reversing it flips every sign after it
Scalar product	$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$	
Geometric form	$\mathbf{a}\cdot\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$	
Angle	$\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$	Divide by **both**
Self product	$\mathbf{a}\cdot\mathbf{a} = \lvert\mathbf{a}\rvert^2$	The identity every geometric proof uses
Vector projection	$\left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b}$	Parallel to $\mathbf{b}$
Scalar projection	$\dfrac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{b}\rvert}$	A signed length
Line	$\mathbf{r} = \mathbf{a} + t\mathbf{d}$	Neither $\mathbf{a}$ nor $\mathbf{d}$ is unique
Line through two points	$\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$	$t = 0$ at A, $t = 1$ at B
Cartesian form of a line	$\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$	Only where no d_i is zero
Sphere	$\lvert\mathbf{r} - \mathbf{c}\rvert = a$	Centre $\mathbf{c}$, radius a
Sphere, expanded	$(x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2 = a^2$	Complete the square to recover it

The identities used in proofs

- $\lvert\mathbf{a} \pm \mathbf{b}\rvert^2 = \lvert\mathbf{a}\rvert^2 \pm 2\,\mathbf{a}\cdot\mathbf{b} + \lvert\mathbf{b}\rvert^2$
- $(\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} - \mathbf{b}) = \lvert\mathbf{a}\rvert^2 - \lvert\mathbf{b}\rvert^2$, which proves the rhombus and the semicircle
- $\lvert\mathbf{a} + \mathbf{b}\rvert^2 + \lvert\mathbf{a} - \mathbf{b}\rvert^2 = 2\big(\lvert\mathbf{a}\rvert^2 + \lvert\mathbf{b}\rvert^2\big)$, the parallelogram law
- Midpoint of AB is $\frac{1}{2}(\mathbf{a} + \mathbf{b})$; the centroid of a triangle is $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$

Classifying two lines · the procedure

Directions	Point test or third component	Conclusion
Parallel	A point of one lies on the other	The **same line**
Parallel	It does not	**Parallel and distinct**
Not parallel	Third component holds	**Intersecting.** Substitute for the point

Directions	Point test or third component	Conclusion
Not parallel	Third component fails	**Skew**

Both halves are needed for skew: not parallel, and no common point. An answer giving only the inconsistent system has proved half of it.

The other standard procedures

Task	The procedure
Is P on the line	Solve for t in one component, check the other two, name the component that fails if it does
Angle between two lines	Angle between the directions, taken acute, so use $\lvert\mathbf{d}_1 \cdot \mathbf{d}_2\rvert$
Foot of the perpendicular from X	Solve $(\mathbf{a} + t\mathbf{d} - \mathbf{x}) \cdot \mathbf{d} = 0$ for t, then substitute
Shortest distance from X to a line	The length of $\overrightarrow{XF}$, then check the scalar product is zero
Centre and radius of a sphere	Complete the square in all three variables
Where a line meets a sphere	Substitute the line, solve the quadratic in t; two roots secant, one tangent, none a miss
Prove a geometric result	Name the vectors, translate the claim, expand using $\mathbf{a} \cdot \mathbf{a} = \lvert\mathbf{a}\rvert^2$

The six ways marks are lost here

1. Writing $\mathbf{a} - \mathbf{b}$ when $\overrightarrow{AB}$ was wanted.
2. Dividing by one magnitude instead of two.
3. Concluding skew without ruling out parallel.
4. Not writing down the third component check.
5. Rounding a surd that the question wanted exact.
6. Starting a geometric proof without naming the vectors.

Not in this course

The cross product (the vector product) and the scalar triple product are not part of Mathematics Extension 2. Every result here is reachable with the scalar product alone.

Where this sits

This document covers focus area 2 of the Year 12 course, Further work with vectors, outcome ME2-12-02.

The full teaching for this material is Chapter 2, Further Work with Vectors of the MA99 Mathematics Extension 2 Year 12 Complete Textbook. This document is the layer between that textbook and the practice papers.

Mathematics Advanced and Mathematics Extension 1 are assumed knowledge for this course and for its examination.

Grounded in the NSW Mathematics Extension 2 11-12 Syllabus (2024), the syllabus in force for the first HSC examination in 2027.

About this resource

MA99 EDUCATION

Scope

NSW Mathematics Extension 2, Year 12. Written to the NSW Mathematics Extension 2 11-12 Syllabus (2024), focus area 2, Further work with vectors (ME2-12-02), for students sitting the HSC in 2027 and later years. Every value, identity and integral was re-derived independently by computer algebra before printing.

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MA99 EDUCATION

Focus area 2 Further work with vectors: Practice Questions

02

NSW MATHEMATICS EXTENSION 2 11-12 SYLLABUS 2024 FOR HSC 2027 AND LATER

How to use this set

Where this sits

This document covers focus area 2 of the Year 12 course, Further work with vectors, outcome ME2-12-02.

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Mathematics Advanced and Mathematics Extension 1 are assumed knowledge for this course and for its examination.

Notation. A vector is set in bold upright here, $\mathbf{a}$, matching the Extension 2 practice papers. By hand you will write it with a tilde underneath, and either is accepted. $\overrightarrow{AB}$ means the vector from A to B, which is $\mathbf{b}-\mathbf{a}$ when $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of A and B from a common origin.

Band	Marks here
Foundation	11
Standard	20
Advanced	24
Exam-style	24

Total 79 marks. Solutions follow the Exam-style band.

Foundation

FOUNDATION One idea, applied once.

Question 1

4 MARKS

Let $\mathbf{a}=(2,-1,3)$ and $\mathbf{b}=(1,4,-2)$.

(a) Find $|\mathbf{a}|$ in exact form. 1 MARK

(b) Find $\mathbf{a}\cdot\mathbf{b}$. 1 MARK

(c) Find the unit vector in the direction of $\mathbf{a}$. 1 MARK

(d) State, with a one line reason, whether the angle between $\mathbf{a}$ and $\mathbf{b}$ is acute or obtuse. 1 MARK

Question 2

2 MARKS

The points A and B have position vectors $(240,70,45)$ and $(600,130,20)$. Find $\overrightarrow{AB}$ and its magnitude in exact form.

Question 3

2 MARKS

Find the value of k for which $(1,k,2)$ is perpendicular to $(2,-1,3)$.

Question 4

1 MARK

Write down a vector equation of the line through $(1,0,2)$ and $(3,4,1)$.

Question 5

2 MARKS

Find the centre and radius of the sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z - 11 = 0$.

STANDARD Two steps, or one step plus a decision.

Question 6

3 MARKS

Find the acute angle between lines with directions $\mathbf{a} = (2, -1, 3)$ and $\mathbf{b} = (1, 4, -2)$, correct to the nearest tenth of a degree.

Question 7

2 MARKS

Determine whether the point $(5, 0, 5)$ lies on the line $\mathbf{r} = (1, 2, 3) + t(2, -1, 1)$. Justify your answer.

Question 8

4 MARKS

Find the vector projection of $\mathbf{a} = (2, -1, 3)$ on $\mathbf{b} = (1, 4, -2)$, and verify that $\mathbf{a}$ minus that projection is perpendicular to $\mathbf{b}$.

Question 9

4 MARKS

Show that the lines $\mathbf{r} = (1, 0, 2) + t(1, 2, -1)$ and $\mathbf{r} = (2, 1, 4) + s(0, 1, -3)$ intersect, and find the point of intersection.

Question 10

3 MARKS

Find the acute angle between the two lines in Question 9, correct to the nearest tenth of a degree.

Question 11

4 MARKS

The points P, Q and R have position vectors $(1, 2, 1)$, $(3, -1, 4)$ and $(-1, 0, 5)$.

(a) Find $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. 1 MARK

(b) Find $\cos(\angle QPR)$ in exact form. 2 MARKS

(c) Hence find $\angle QPR$ to the nearest tenth of a degree. 1 MARK

ADVANCED Three or more steps, or a method you must choose first.

Question 12

4 MARKS

Show that the lines $\mathbf{r} = (2, 0, 1) + t(1, 2, 0)$ and $\mathbf{r} = (1, 3, 0) + s(0, 1, 1)$ are skew. Your answer must establish both conditions required.

Question 13

5 MARKS

Find the shortest distance from the point $X(4, 5, 1)$ to the line $\mathbf{r} = (1, 0, 2) + t(1, 2, -1)$, in exact form. Verify your foot of the perpendicular.

Question 14

4 MARKS

Find the points where the line $\mathbf{r} = t(1, 1, 1)$ meets the sphere $(x - 2)^2 + (y + 1)^2 + (z - 3)^2 = 25$.

Question 15

4 MARKS

$\mathbf{a}$ and $\mathbf{b}$ are non zero vectors with $|\mathbf{a}| = |\mathbf{b}|$. Prove that $\mathbf{a} + \mathbf{b}$ is perpendicular to $\mathbf{a} - \mathbf{b}$, and state the geometric result this establishes.

Question 16

7 MARKS

Prove, using vectors, that the medians of a triangle are concurrent, and that the point of concurrency divides each median in the ratio $2 : 1$.

EXAM-STYLE Multi-part, written the way the HSC writes them.

Question 17

12 MARKS

(a) In the Wollundry survey frame, Station One is the origin, Station Two is at $A(240, 70, 45)$ and Station Three is at $B(600, 130, 20)$, in metres. Find the exact length of the haul cable from Station Two to Station Three. 2 MARKS

(b) Find the exact direct distance from Station One to Station Three, and hence show that routing the cable via Station Two costs about 5.6 m of extra cable. 3 MARKS

(c) Find the size of $\angle OAB$, to the nearest tenth of a degree, and explain what its value tells you about how nearly straight the two leg route is. 4 MARKS

(d) A survey mast has its base at $(420, 100, 35)$. Determine whether the base lies on the straight haul cable from A to B, naming the component that settles it. 3 MARKS

Question 18

12 MARKS

(a) Prove that for any vectors $\mathbf{a}$ and $\mathbf{b}$, $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + 2\,\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2$. 2 MARKS

(b) Hence prove the parallelogram law, that $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2\big(|\mathbf{a}|^2 + |\mathbf{b}|^2\big)$. 2 MARKS

(c) $OACB$ is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. Given that the diagonals have equal length, prove that $OACB$ is a rectangle. 4 MARKS

(d) Prove that the angle in a semicircle is a right angle. 4 MARKS

WORKING SPACE

Worked solutions · Foundation

1

(a) $|\mathbf{a}| = \sqrt{4+1+9} = \sqrt{14}$.

(b) $\mathbf{a}\cdot\mathbf{b} = (2)(1) + (-1)(4) + (3)(-2) = 2 - 4 - 6 = -8$.

(c) $\hat{\mathbf{a}} = \dfrac{1}{\sqrt{14}}(2, -1, 3)$.

(d) Obtuse, because $\mathbf{a}\cdot\mathbf{b} = -8 < 0$ and the magnitudes are positive, so $\cos\theta < 0$.

Part (d) needs no magnitudes at all. The sign of the scalar product settles acute against obtuse on its own.

2

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (360, 60, -25)$.

$$|\overrightarrow{AB}| = \sqrt{129600 + 3600 + 625} = \sqrt{133825} = 5\sqrt{5353}.$$

WHERE THE MARKS ARE	
1 mark	$\overrightarrow{AB}$ in the correct order
1 mark	Magnitude, simplified to $5\sqrt{5353}$

$133825 = 25 \times 5353$, and $5353 = 53 \times 101$ has no square factor, so $5\sqrt{5353}$ is fully simplified.

3

Perpendicular means the scalar product vanishes:

$$(1)(2) + (k)(-1) + (2)(3) = 0 \implies 2 - k + 6 = 0 \implies k = 8.$$

WHERE THE MARKS ARE	
1 mark	Sets the scalar product to zero
1 mark	$k = 8$

4

The direction is $(3, 4, 1) - (1, 0, 2) = (2, 4, -1)$, so

$$\mathbf{r} = (1, 0, 2) + t(2, 4, -1).$$

Any point on the line and any non zero multiple of that direction is equally correct. $\mathbf{r} = (3, 4, 1) + t(-2, -4, 1)$ is the same line.

5

Complete the square in each variable:

$$(x^2 - 4x) + (y^2 + 2y) + (z^2 - 6z) = 11$$

$$(x - 2)^2 + (y + 1)^2 + (z - 3)^2 = 11 + 4 + 1 + 9 = 25$$

Centre $(2, -1, 3)$, radius 5.

WHERE THE MARKS ARE	
1 mark	Completes the square in all three variables
1 mark	Centre and radius

Worked solutions · Standard

6

$$\mathbf{a} \cdot \mathbf{b} = -8, \qquad |\mathbf{a}| = \sqrt{14}, \qquad |\mathbf{b}| = \sqrt{21}$$

$$\cos\theta = \frac{-8}{\sqrt{14}\sqrt{21}} = \frac{-8}{\sqrt{294}} = \frac{-8}{7\sqrt{6}} \quad \text{since } 294 = 49 \times 6$$

$$\theta = \cos^{-1}\left(\frac{-8}{7\sqrt{6}}\right) \approx 117.8^\circ$$

The question asks for the **acute** angle, so take the supplement: $180^\circ - 117.8^\circ = 62.2^\circ$.

WHERE THE MARKS ARE	
1 mark	Scalar product and both magnitudes
1 mark	Correct cosine, and the angle between the vectors
1 mark	Recognises the question asked for the acute angle and takes the supplement

This is the trap in the question. The angle between the two vectors is obtuse; the acute angle between the directions is its supplement. Read which was asked.

7

The x component gives $1 + 2t = 5$, so $t = 2$. Checking the others at $t = 2$: the y component is $2 - 2 = 0$ and the z component is $3 + 2 = 5$. Both agree with $(5, 0, 5)$, so the point does lie on the line.

WHERE THE MARKS ARE	
1 mark	Solves for t in one component
1 mark	Checks the remaining two and concludes

One component agreeing proves nothing. The mark is for the check.

8

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\,\mathbf{b} = \frac{-8}{21}(1, 4, -2) \quad \text{since b·b = 1 + 16 + 4 = 21}$$

$$= \left(-\tfrac{8}{21},\ -\tfrac{32}{21},\ \tfrac{16}{21}\right)$$

Then

$$\mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a} = \left(2 + \tfrac{8}{21},\ -1 + \tfrac{32}{21},\ 3 - \tfrac{16}{21}\right) = \left(\tfrac{50}{21},\ \tfrac{11}{21},\ \tfrac{47}{21}\right),$$

and its scalar product with $\mathbf{b}$ is

$$\tfrac{1}{21}\left[(50)(1) + (11)(4) + (47)(-2)\right] = \tfrac{1}{21}\left[50 + 44 - 94\right] = 0,$$

confirming the perpendicularity.

WHERE THE MARKS ARE	
1 mark	Correct $\mathbf{b}\cdot\mathbf{b} = 21$
1 mark	Correct vector projection
1 mark	Computes $\mathbf{a}$ minus the projection
1 mark	Shows the scalar product with $\mathbf{b}$ is zero

Dividing by $\mathbf{b}\cdot\mathbf{b}$, not by $|\mathbf{b}|$, is what makes the result a vector parallel to $\mathbf{b}$. Dividing by $|\mathbf{b}|$ gives the scalar projection instead.

9

The directions $(1, 2, -1)$ and $(0, 1, -3)$ are not scalar multiples of one another, since matching the first components would need the factor 0 and the second would not. So the lines are not parallel. Equating components:

$x:\ 1+t=2 \implies t=1$

$y:\ 2t=1+s \implies s=1$

$z:\ 2-t=4-3s \implies 1=1\ \checkmark$ the check holds

So the lines intersect. Substituting $t=1$ into the first line gives $(2,2,1)$, and $s=1$ in the second gives $(2,2,1)$ as well.

WHERE THE MARKS ARE	
1 mark	Solves for t
1 mark	Solves for s
1 mark	Substitutes into the third component and shows it holds
1 mark	States the point of intersection

10

$\mathbf{d}_1 \cdot \mathbf{d}_2 = (1)(0)+(2)(1)+(-1)(-3) = 5$

$|\mathbf{d}_1| = \sqrt{6}, \qquad |\mathbf{d}_2| = \sqrt{10}$

$\cos\theta = \dfrac{5}{\sqrt{60}} = \dfrac{5}{2\sqrt{15}}$

$\theta = \cos^{-1}\left(\dfrac{5}{2\sqrt{15}}\right) \approx 49.8^\circ$

The cosine is positive, so this angle is already acute and no supplement is needed.

WHERE THE MARKS ARE	
1 mark	Scalar product of the directions
1 mark	Both magnitudes and the cosine
1 mark	Angle to the required accuracy

11

(a) $\overrightarrow{PQ} = (2,-3,3)$ and $\overrightarrow{PR} = (-2,-2,4)$.

(b)

$\overrightarrow{PQ} \cdot \overrightarrow{PR} = (2)(-2)+(-3)(-2)+(3)(4) = -4+6+12 = 14$

$\left|\overrightarrow{PQ}\right| = \sqrt{4+9+9} = \sqrt{22}, \qquad \left|\overrightarrow{PR}\right| = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$

$\cos(\angle QPR) = \dfrac{14}{\sqrt{22}\cdot 2\sqrt{6}} = \dfrac{14}{2\sqrt{132}} = \dfrac{7}{2\sqrt{33}}$ since $\sqrt{132} = 2\sqrt{33}$

(c) $\angle QPR = \cos^{-1}\left(\dfrac{7}{2\sqrt{33}}\right) \approx 52.5^\circ.$

WHERE THE MARKS ARE	
1 mark	(a) both vectors, in the right order
1 mark	(b) scalar product and both magnitudes
1 mark	(b) exact cosine, simplified
1 mark	(c) the angle to one decimal place

Worked solutions · Advanced

12

Condition one, not parallel. The directions are $(1, 2, 0)$ and $(0, 1, 1)$. For these to be parallel we would need $(1, 2, 0) = \lambda(0, 1, 1)$, and the first component forces $1 = 0$, which is impossible. So the lines are not parallel.

Condition two, no common point. Suppose they meet. Equating components:

$$x:\ 2 + t = 1 \implies t = -1$$

$$y:\ 2t = 3 + s \implies s = 2t - 3 = -5$$

$$z:\ \text{on } L_1,\ 1; \qquad \text{on } L_2,\ 0 + s = -5$$

These are not equal, so the third equation fails and no values of t and s satisfy all three. Hence the lines do not intersect.

The lines are neither parallel nor intersecting, so they are **skew**.

WHERE THE MARKS ARE	
1 mark	Shows the directions are not parallel, with a reason
1 mark	Solves two components for t and s
1 mark	Substitutes into the third and shows it fails
1 mark	Concludes skew, citing both conditions

A candidate who solves all three at once, finds no solution and writes therefore skew has one of the four marks at risk, because parallel distinct lines also give no solution.

13

Let F be the foot of the perpendicular, at parameter t. Then

$$\overrightarrow{XF} = (1 + t, 2t, 2 - t) - (4, 5, 1) = (t - 3,\ 2t - 5,\ 1 - t)$$

$$\overrightarrow{XF} \cdot (1, 2, -1) = (t - 3) + 2(2t - 5) - (1 - t) = 6t - 14 \quad \text{expand and collect}$$

$$6t - 14 = 0 \implies t = \tfrac{7}{3}$$

So $F = \left(\tfrac{10}{3}, \tfrac{14}{3}, -\tfrac{1}{3}\right)$ and

$$\overrightarrow{XF} = \left(-\tfrac{2}{3},\ -\tfrac{1}{3},\ -\tfrac{4}{3}\right), \qquad \left|\overrightarrow{XF}\right| = \tfrac{1}{3}\sqrt{4 + 1 + 16} = \frac{\sqrt{21}}{3}.$$

Verification. $\overrightarrow{XF} \cdot (1, 2, -1) = -\tfrac{2}{3} - \tfrac{2}{3} + \tfrac{4}{3} = 0$, so F is indeed the foot.

WHERE THE MARKS ARE	
1 mark	Writes the joining vector in terms of t
1 mark	Sets its scalar product with the direction to zero
1 mark	$t = \frac{7}{3}$
1 mark	Exact distance $\frac{\sqrt{21}}{3}$
1 mark	Verification that the scalar product is zero

The verification is a mark here because the question asked for it, and it is worth doing even when it is not asked: a slip in t shows up immediately.

14

A point of the line is (t, t, t). Substituting:

$$(t-2)^2 + (t+1)^2 + (t-3)^2 = 25$$

$$t^2 - 4t + 4 + t^2 + 2t + 1 + t^2 - 6t + 9 = 25 \quad \text{expand}$$

$$3t^2 - 8t + 14 = 25 \implies 3t^2 - 8t - 11 = 0 \quad \text{collect}$$

$$(3t - 11)(t + 1) = 0 \implies t = \tfrac{11}{3},\ t = -1 \quad \text{factorise}$$

The two points are $\left(\frac{11}{3}, \frac{11}{3}, \frac{11}{3}\right)$ and $(-1, -1, -1)$.

WHERE THE MARKS ARE	
1 mark	Substitutes the line into the sphere
1 mark	Reaches $3t^2 - 8t - 11 = 0$
1 mark	Both values of t
1 mark	Both points

Two distinct real roots means the line is a secant. A repeated root would mean a tangent, and a negative discriminant means that the line misses the sphere entirely.

15

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} \quad \text{expand}$$

$$= |\mathbf{a}|^2 - |\mathbf{b}|^2 \quad \text{the cross terms cancel, since the product is commutative}$$

$$= 0 \quad \text{because the magnitudes are equal}$$

Both $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are non zero: the first vanishes only if $\mathbf{b} = -\mathbf{a}$ and the second only if $\mathbf{a} = \mathbf{b}$, and in either degenerate case there is no parallelogram. A zero scalar product of non zero vectors means they are perpendicular.

The geometric result. In a parallelogram with two adjacent sides of equal length, that is a **rhombus**, the diagonals $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular.

WHERE THE MARKS ARE	
1 mark	Expands the product
1 mark	Notes the cross terms cancel by commutativity
1 mark	Uses $\lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert$ to get zero
1 mark	Names the geometric result, the diagonals of a rhombus

16

Let the vertices A, B, C have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ from any origin.

The midpoint of BC is $M_A = \frac{1}{2}(\mathbf{b}+\mathbf{c})$, so the median from A is

$$\mathbf{r} = \mathbf{a} + t\left(\tfrac{1}{2}(\mathbf{b}+\mathbf{c}) - \mathbf{a}\right).$$

At $t = \frac{2}{3}$,

$$\mathbf{r} = \mathbf{a} - \tfrac{2}{3}\mathbf{a} + \tfrac{1}{3}(\mathbf{b}+\mathbf{c}) = \tfrac{1}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{b} + \tfrac{1}{3}\mathbf{c} = \tfrac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}) = \mathbf{g}.$$

The expression $\mathbf{g}$ is symmetric in $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. The median from B is obtained from the median from A by exchanging the labels $\mathbf{a}$ and $\mathbf{b}$, and the same computation therefore yields the same point $\mathbf{g}$; likewise for the median from C. So all three medians pass through $\mathbf{g}$ and are concurrent.

The ratio. On the median from A, $t = 0$ is at A and $t = 1$ is at M_A, and $\mathbf{g}$ occurs at $t = \frac{2}{3}$. So $\mathbf{g}$ divides AM_A in the ratio $\frac{2}{3} : \frac{1}{3}$, that is $2:1$, measured from the vertex.

WHERE THE MARKS ARE	
1 mark	Names position vectors and the midpoint of one side
1 mark	Writes the vector equation of one median
2 marks	Evaluates at $t = \frac{2}{3}$ and simplifies to $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$
2 marks	Argues from symmetry that the other two medians pass through the same point, and concludes concurrency
1 mark	Reads the $2:1$ ratio off the parameter, from the vertex

The symmetry argument is the whole economy of this proof. It is legitimate, but you must say **why** the expression is symmetric, which is that exchanging two labels leaves $\mathbf{a}+\mathbf{b}+\mathbf{c}$ unchanged.

Worked solutions · Exam-style

17

(a) $\overrightarrow{AB} = (360, 60, -25)$, so

$$\left|\overrightarrow{AB}\right| = \sqrt{129600 + 3600 + 625} = \sqrt{133825} = 5\sqrt{5353} \approx 365.82 \text{ m}.$$

(b) $\overrightarrow{OB} = (600, 130, 20)$, so

$$\left|\overrightarrow{OB}\right| = \sqrt{360000 + 16900 + 400} = \sqrt{377300} = 70\sqrt{77} \approx 614.25 \text{ m}.$$

Also $\left|\overrightarrow{OA}\right| = \sqrt{64525} = 5\sqrt{2581} \approx 254.02$ m. The two leg route measures $254.02 + 365.82 = 619.84$ m, and

$$619.84 - 614.25 = 5.59 \text{ m},$$

so the dogleg costs about 5.6 m of extra cable.

(c) At A the two directions are $\overrightarrow{AO} = (-240, -70, -45)$ and $\overrightarrow{AB} = (360, 60, -25)$.

$$\overrightarrow{AO} \cdot \overrightarrow{AB} = -86400 - 4200 + 1125 = -89475$$

$$\left|\overrightarrow{AO}\right| = 5\sqrt{2581}, \qquad \left|\overrightarrow{AB}\right| = 5\sqrt{5353}$$

$$\cos(\angle OAB) = \frac{-89475}{25\sqrt{2581}\sqrt{5353}} \approx -0.96286$$

$$\angle OAB \approx 164.3^\circ$$

The angle is close to 180°, which says the three stations are very nearly collinear: the path bends by only about 15.7° at Station Two. That is the geometric reason the extra cable in part (b) is only 5.6 m out of 620 m, which is under one per cent.

(d) On the cable, $\mathbf{r} = (240, 70, 45) + t(360, 60, -25)$. The x component gives $240 + 360t = 420$, so $t = \frac{1}{2}$. The y component at $t = \frac{1}{2}$ is $70 + 30 = 100$, which agrees. The z component at $t = \frac{1}{2}$ is

$$45 - 12.5 = 32.5 \neq 35,$$

so the z component fails and the mast base does **not** lie on the cable. It sits 2.5 m above it.

WHERE THE MARKS ARE	
2 marks	(a) $\overrightarrow{AB}$ and its exact magnitude
2 marks	(b) the direct distance, exact and approximate
1 mark	(b) the comparison, giving about 5.6 m
2 marks	(c) scalar product and both magnitudes
1 mark	(c) the angle, 164.3 degrees
1 mark	(c) interprets it as near collinearity, linked to part (b)
2 marks	(d) solves for t and tests the other components
1 mark	(d) names the z component as the one that fails

Part (c) is four marks and one of them is the sentence in the last paragraph. An answer that computes 164.3° and stops has not answered what the value tells you.

18

(a) Using $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$ with $\mathbf{v} = \mathbf{a} + \mathbf{b}$:

$$|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + 2\,\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = |\mathbf{a}|^2 + 2\,\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2,$$

where the two middle terms combine because the scalar product is commutative.

(b) Replacing $\mathbf{b}$ by $-\mathbf{b}$ in part (a) gives

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 - 2\,\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2.$$

Adding the two results, the scalar product terms cancel:

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2 = 2\big(|\mathbf{a}|^2 + |\mathbf{b}|^2\big).$$

(c) The diagonals of $OACB$ are $\overrightarrow{OC} = \mathbf{a} + \mathbf{b}$ and $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$. Equal length means

$$|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}-\mathbf{b}|^2$$

$$|\mathbf{a}|^2 + 2\,\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 = |\mathbf{a}|^2 - 2\,\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 \quad \text{by part (a) and its variant}$$

$$4\,\mathbf{a}\cdot\mathbf{b} = 0 \implies \mathbf{a}\cdot\mathbf{b} = 0$$

$\mathbf{a}$ and $\mathbf{b}$ are non zero, being sides of a parallelogram, so they are perpendicular. A parallelogram with a right angle at one vertex has right angles at all four, so $OACB$ is a rectangle.

(d) Take the centre of the circle as origin and let the diameter join A with position vector $\mathbf{a}$ to A' with position vector $-\mathbf{a}$. Let P, with position vector $\mathbf{p}$, be any point of the circle other than A and A'. Since all three lie on the circle of radius r, $|\mathbf{p}| = |\mathbf{a}| = r$.

$$\overrightarrow{AP}\cdot\overrightarrow{A'P} = (\mathbf{p}-\mathbf{a})\cdot(\mathbf{p}+\mathbf{a}) = |\mathbf{p}|^2 - |\mathbf{a}|^2 = r^2 - r^2 = 0.$$

Neither chord is the zero vector, since P is neither A nor A', so the two chords are perpendicular and the angle at P is a right angle.

WHERE THE MARKS ARE

2 marks	(a) expands using $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$ and notes commutativity
2 marks	(b) obtains the minus version and adds
2 marks	(c) identifies both diagonals and equates their squared lengths
2 marks	(c) deduces $\mathbf{a}\cdot\mathbf{b} = 0$ and concludes rectangle
2 marks	(d) sets the centre as origin, with the diameter as $\pm\mathbf{a}$
2 marks	(d) expands to $\|\mathbf{p}\|^2 - \|\mathbf{a}\|^2 = 0$ and concludes

Parts (c) and (d) are the same algebra as Question 15, run in two different directions: (c) starts from equal diagonals and derives perpendicular sides, and (d) starts from equal radii and derives a perpendicular pair. Recognising that saves you the thinking time in the examination.

About this resource

MA99 EDUCATION

Scope

NSW Mathematics Extension 2, Year 12. Written to the NSW Mathematics Extension 2 11-12 Syllabus (2024), focus area 2, Further work with vectors (ME2-12-02), for students sitting the HSC in 2027 and later years. Questions that need a setting use the Wollundry Gorge cableway, an invented aerial cableway, and every value, identity and integral was re-derived independently by computer algebra before printing.

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